(c) Use these results to prove that the *discrete fractional integral operator*

$$\{a_j\}_{j\in\mathbf{Z}^n} \mapsto \Big\{ \sum_{k\in\mathbf{Z}^n} \frac{a_k}{(|j-k|+1)^{n-\alpha}} \Big\}_{j\in\mathbf{Z}^n}$$

maps $\ell^s(\mathbf{Z}^n)$ to $\ell^t(\mathbf{Z}^n)$ when $0<\alpha<n$, $1<s<t$, and $\frac{1}{s}-\frac{1}{t}=\frac{\alpha}{n}$.

1.2.11. Show that the operator,

$$\mathcal{I}_{\alpha,\alpha}(f)(x_1,x_2) = \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} f(x_1-y_1,x_2-y_2)|y_1|^{-n+\alpha}|y_2|^{-n+\alpha}dy_1dy_2\,,$$

acting on Schwartz functions f on $\mathbf{R}^{2n}$, maps $L^p(\mathbf{R}^{2n})$ to $L^q(\mathbf{R}^{2n})$ whenever $0<\alpha<n$, $\frac{\alpha}{n}+\frac{1}{q}=\frac{1}{p}$, and $1<p<q<\infty$.

[*Hint:* Write $\mathcal{I}_{\alpha,\alpha}(f)(x_1,x_2)=c_0\mathcal{I}_\alpha^{(1)}\big(\mathcal{I}_\alpha^{(2)}(f)(x_2)\big)(x_1)$, where $\mathcal{I}_\alpha^{(2)}$ is a fractional integral operator acting in the x_2 variable of the function $f(x_1,x_2)$ with x_1 frozen, and $\mathcal{I}_\alpha^{(1)}$ is defined analogously.]

1.2.12. Fill in the following steps to provide an alternative proof of Theorem 1.2.3 when $p=1$. Without loss of generality assume that f is nonnegative and smooth, has compact support, and satisfies $\|f\|_{L^1}=1$. Let $E_\lambda=\{x\in\mathbf{R}^n:\ \mathcal{I}_s(f)(x)>\lambda\}$ for $\lambda>0$. Prove that

$$\mathcal{I}_s(f)(x)\le\sum_{j\in\mathbf{Z}}2^{(j-1)(s-n)}\int_{|y|\le 2^j}f(x-y)\,dy,$$

and that

$$\int_{E_\lambda}\int_{|y|\le 2^j}f(x-y)dy\,dx\le\min(|E_\lambda|,v_n2^{jn})\,.$$

Using these facts and $|E_\lambda|\le\frac{1}{\lambda}\int_{E_\lambda}\mathcal{I}_s(f)(x)\,dx$ conclude that $|E_\lambda|\le C(n,s)\frac{1}{\lambda}|E_\lambda|^{\frac{s}{n}}$.

1.3 Sobolev Spaces

In this section we study a quantitative way of measuring the smoothness of functions. Sobolev spaces serve exactly this purpose. They measure the smoothness of functions in terms of the integrability of their derivatives. We begin with the classical definition of Sobolev spaces.

Definition 1.3.1. Let k be a nonnegative integer, and let $1<p<\infty$. The *Sobolev space* $L^p_k(\mathbf{R}^n)$ is defined as the space of functions f in $L^p(\mathbf{R}^n)$ all of whose distributional derivatives $\partial^\alpha f$ are also in $L^p(\mathbf{R}^n)$ for all multi-indices α that satisfy $|\alpha|\le k$. This space is normed by the expression

$$\|f\|_{L^p_k}=\sum_{|\alpha|\le k}\|\partial^\alpha f\|_{L^p}, \tag{1.3.1}$$

where $\partial^{(0,\dots,0)}f=f$.