Definition 3.5.3. We define the *Orlicz maximal operator*

$$M_{L\log(e+L)}(f)(x) = \sup_{Q \ni x} \|f\|_{L\log(e+L)(Q, \frac{dx}{|Q|})},$$

where the supremum is taken over all cubes Q with sides parallel to the axes that contain the given point x and the norm is taken with respect to normalized Lebesgue measure on Q.

The boundedness properties of this maximal operator are a consequence of the following lemma.

Lemma 3.5.4. *There is a positive constant $c(n)$ such that for any cube Q in $\mathbf{R}^n$ and any nonnegative locally integrable function w, we have*

$$\|w\|_{L\log(e+L)(Q)} \le \frac{c(n)}{|Q|} \int_Q M_c(w)\,dx\,, \tag{3.5.2}$$

where M_c is the Hardy–Littlewood maximal operator with respect to cubes. Hence, for some other dimensional constant $c'(n)$ and all nonnegative w in $L^1_{\text{loc}}(\mathbf{R}^n)$ the inequality

$$M_{L\log(e+L)}(w)(x) \le c'(n) M^2(w)(x) \tag{3.5.3}$$

is valid, where $M^2 = M \circ M$ and M is the Hardy–Littlewood maximal operator.

Proof. Fix a cube Q in $\mathbf{R}^n$ with sides parallel to the axes. We introduce a *maximal operator associated with Q* as follows:

$$M_c^Q(f)(x) = \sup_{\substack{R \ni x \\ R \subseteq Q}} \frac{1}{|R|} \int_R |f(y)|\,dy\,,$$

where the supremum is taken over cubes R in $\mathbf{R}^n$ with sides parallel to the axes. The key estimate follows from the following local version of the reverse weak-type $(1,1)$ estimate of the Hardy–Littlewood maximal function (see Exercise 2.1.4(c) in [156]). For each nonnegative function f on $\mathbf{R}^n$ and $\alpha \ge \operatorname{Avg}_Q f$, we have

$$\frac{1}{\alpha} \int_{Q \cap \{f > \alpha\}} f\,dx \le 2^n \left|\{x \in Q :\ M_c^Q(f)(x) > \alpha\}\right|. \tag{3.5.4}$$

Indeed, to prove (3.5.4), we apply Proposition 2.1.20 in [156] to the function f and the number $\alpha > 0$. Then there exists a collection of disjoint (possibly empty) open cubes Q_j such that for almost all $x \in \big(\bigcup_j Q_j\big)^c$ we have $f(x) \le \alpha$ and

$$\alpha < \frac{1}{|Q_j|} \int_{Q_j} f(t)\,dt \le 2^n \alpha\,. \tag{3.5.5}$$

According to Corollary 2.1.21 in [156] we have $|Q \setminus (\bigcup_j Q_j)| \subseteq |\{f \le \alpha\}|$. This implies that $|Q \cap \{f > \alpha\}| \le \big|\bigcup_j Q_j\big|$, which is at most $|\{x \in Q :\ M_c^Q(f)(x) > \alpha\}|$. Multiplying both sides of (3.5.5) by $|Q_j|/\alpha$ and summing over j we obtain (3.5.4).