3.4.3 Interpolation Using *BMO*

We continue this section by proving an interpolation result in which the space L^∞ is replaced by BMO. The sharp function plays a key role in the following theorem.

Theorem 3.4.7. *Let $1 \le p_0 < \infty$. Let T be a linear operator that maps $L^{p_0}(\mathbf{R}^n)$ to $L^{p_0}(\mathbf{R}^n)$ with bound A_0, and $L^\infty(\mathbf{R}^n)$ to $BMO(\mathbf{R}^n)$ with bound A_1. Then for all p with $p_0 < p < \infty$ there is a constant C_{n,p,p_0} such that for all $f \in L^p$ we have*

$$\big\|T(f)\big\|_{L^p(\mathbf{R}^n)} \le C_{n,p,p_0} A_0^{\frac{p_0}{p}} A_1^{1-\frac{p_0}{p}} \big\|f\big\|_{L^p(\mathbf{R}^n)} . \tag{3.4.12}$$

Remark 3.4.8. In certain applications, the operator T may not be a priori defined on all of $L^{p_0} + L^\infty$ but only on some subspace of it. In this case one may state that the hypotheses and the conclusion of the preceding theorem hold for a subspace of these spaces.

Proof. We consider the operator

$$S(f) = M^\#(T(f))$$

defined for $f \in L^{p_0} + L^\infty$. It is easy to see that S is a sublinear operator. We prove that S maps L^∞ to itself and L^{p_0} to itself if $p_0 > 1$ or L^1 to $L^{1,\infty}$ if $p_0 = 1$. For $f \in L^{p_0}$ we have

$$\begin{aligned}\big\|S(f)\big\|_{L^{p_0}} &= \big\|M^\#(T(f))\big\|_{L^{p_0}} \le 2\big\|M_c(T(f))\big\|_{L^{p_0}} \\ &\le C_{n,p_0}\big\|T(f)\big\|_{L^{p_0}} \le C_{n,p_0} A_0 \big\|f\big\|_{L^{p_0}} ,\end{aligned}$$

where the three L^{p_0} norms on the top line should be replaced by $L^{1,\infty}$ if $p_0 = 1$. For $f \in L^\infty$ one has

$$\big\|S(f)\big\|_{L^\infty} = \big\|M^\#(T(f))\big\|_{L^\infty} = \big\|T(f)\big\|_{BMO} \le A_1 \big\|f\big\|_{L^\infty} .$$

Interpolating between these estimates using Theorem 1.3.2 in [156], we deduce

$$\big\|M^\#(T(f))\big\|_{L^p} = \big\|S(f)\big\|_{L^p} \le C_{p,p_0} A_0^{\frac{p_0}{p}} A_1^{1-\frac{p_0}{p}} \big\|f\big\|_{L^p}$$

for all $f \in L^p$, where $p_0 < p < \infty$.

Consider now a function $h \in L^p \cap L^{p_0}$. In the case $p_0 > 1$, $M_d(T(h)) \in L^{p_0}$; hence Corollary 3.4.6 is applicable and gives

$$\big\|T(h)\big\|_{L^p} \le C_n(p)\, C_{p,p_0} A_0^{\frac{p_0}{p}} A_1^{1-\frac{p_0}{p}} \big\|h\big\|_{L^p} .$$

Density yields the same estimate for all $f \in L^p(\mathbf{R}^n)$. If $p_0 = 1$, one applies the same idea but needs the endpoint estimate of Exercise 3.4.6, since $M_d(T(h)) \in L^{1,\infty}$. $\square$

3.4.4 Estimates for Singular Integrals Involving the Sharp Function

We use the sharp function to obtain pointwise estimates for singular integrals. These enable us to recover previously obtained estimates for singular integrals, but also to deduce a new endpoint boundedness result from L^∞ to BMO.

We recall some facts about singular integral operators. Suppose that K is a function defined on $\mathbf{R}^n \setminus \{0\}$ that satisfies

$$|K(x)| \le A_1 |x|^{-n}, \tag{3.4.13}$$

$$|K(x-y) - K(x)| \le A_2 |y|^\delta \, |x|^{-n-\delta} \qquad \text{when } |x| \ge 2|y| > 0, \tag{3.4.14}$$

$$\sup_{r<R<\infty} \left| \int_{r\le |x| \le R} K(x)\, dx \right| \le A_3 . \tag{3.4.15}$$

Let W be a tempered distribution that coincides with K on $\mathbf{R}^n \setminus \{0\}$ and let T be the linear operator given by convolution with W.

Under these assumptions we have that T is L^2 bounded with norm at most a constant multiple of $A_1 + A_2 + A_3$ (Theorem 5.4.1 in [156]), and hence it is also L^p bounded with a similar norm on L^p for $1 < p < \infty$ (Theorem 5.3.3 in [156]).

Theorem 3.4.9. *Let T be given by convolution with a distribution W that coincides with a function K on $\mathbf{R}^n \setminus \{0\}$ satisfying (3.4.14). Assume that T has an extension that is L^2 bounded with a norm B. Then there is a constant C_n such that for any $s > 1$ the estimate*

$$M^\#(T(f))(x) \le C_n (A_2 + B) \max(s, (s-1)^{-1}) M(|f|^s)^{\frac{1}{s}}(x) \tag{3.4.16}$$

is valid for all f in $\bigcup_{s \le p < \infty} L^p$ and all $x \in \mathbf{R}^n$.

Proof. In view of Proposition 3.4.2 (2), given any cube Q, it suffices to find a constant a_Q such that

$$\frac{1}{|Q|} \int_Q |T(f)(y) - a_Q| \, dy \le C_n \max(s, (s-1)^{-1}) (A_2 + B) M(|f|^s)^{\frac{1}{s}}(x) \tag{3.4.17}$$

for almost all $x \in Q$. To prove this estimate we employ a well-known theme. We write $f = f_Q^0 + f_Q^\infty$, where $f_Q^0 = f \chi_{6\sqrt{n}Q}$ and $f_Q^\infty = f \chi_{(6\sqrt{n}Q)^c}$. Here $Q^* = 6\sqrt{n}Q$ denotes the cube that is concentric with Q, has sides parallel to those of Q, and has side length $6\sqrt{n}\,\ell(Q)$, where $\ell(Q)$ is the side length of Q.

We now fix an f in $\bigcup_{s \le p < \infty} L^p$ and we select $a_Q = T(f_Q^\infty)(x)$. Then a_Q is finite (and thus well defined) for all $x \in Q$. Indeed, for all $x \in Q$, (3.4.13) yields

$$|T(f_Q^\infty)(x)| = \left| \int_{(Q^*)^c} f(y) K(x-y) \, dy \right| \le \|f\|_{L^p} \left(\int_{|x-y| \ge c_n \ell(Q)} \frac{A_1^{p'} \, dy}{|x-y|^{np'}} \right)^{\frac{1}{p'}} < \infty ,$$