$$\frac{w\big(\{x \in Q_j:\ M_d(f)(x) > 2\lambda,\ M^\#(f)(x) \le \gamma\lambda\}\big)}{w(Q_j)} \le C_2\,(2^n\gamma)^{\varepsilon_0}\,.$$

Summing over j we obtain (3.4.3) with $C = C_2\,2^{n\varepsilon_0}$. □

Good lambda inequalities can be used to obtain L^p bounds for quantities they contain. For example, we use Theorem 3.4.4 to obtain the equivalence of the L^p norms of $M_d(f)$ and $M^\#(f)$. Since $M^\#(f)$ is pointwise controlled by $2M_c(f)$ and

$$\big\|M_c(f)\big\|_{L^p} \le C(p,n)\big\|f\big\|_{L^p} \le C(p,n)\big\|M_d(f)\big\|_{L^p},$$

we have the estimate

$$\big\|M^\#(f)\big\|_{L^p(\mathbf{R}^n)} \le 2C(p,n)\big\|M_d(f)\big\|_{L^p(\mathbf{R}^n)}$$

for all f in $L^p(\mathbf{R}^n)$. The next theorem says that the converse estimate is valid.

Theorem 3.4.5. *Let $0 < p_0 \le p < \infty$. Then there is a constant $C_n(p)$ such that for all functions f in $L^1_{\mathrm{loc}}(\mathbf{R}^n)$ with $M_d(f) \in L^{p_0}(\mathbf{R}^n)$ we have*

$$\big\|M_d(f)\big\|_{L^p(\mathbf{R}^n)} \le C_n(p)\big\|M^\#(f)\big\|_{L^p(\mathbf{R}^n)}\,. \tag{3.4.9}$$

If $w \in A_q$ for some q satisfying $1 \le q \le \infty$, there is a constant $C_n(p,q,[w]_{A_q})$, such that if $M_d(f) \in L^{p_0}(\mathbf{R}^n, w)$, then

$$\big\|M_d(f)\big\|_{L^p(\mathbf{R}^n,w)} \le C_n(p,q,[w]_{A_q})\big\|M^\#(f)\big\|_{L^p(\mathbf{R}^n,w)}\,. \tag{3.4.10}$$

Proof. Fix $p \ge p_0$ with $p < \infty$. For a positive real number N we set

$$I_N = \int_0^N p\lambda^{p-1}\big|\{x \in \mathbf{R}^n:\ M_d(f)(x) > \lambda\}\big|\,d\lambda\,.$$

We note that I_N is finite, since $p \ge p_0$ and it is bounded by

$$\frac{pN^{p-p_0}}{p_0}\int_0^N p_0\lambda^{p_0-1}\big|\{x \in \mathbf{R}^n : M_d(f)(x) > \lambda\}\big|\,d\lambda \le \frac{pN^{p-p_0}}{p_0}\big\|M_d(f)\big\|_{L^{p_0}}^{p_0} < \infty\,.$$

We now write

$$I_N = 2^p\int_0^{\frac{N}{2}} p\lambda^{p-1}\big|\{x \in \mathbf{R}^n:\ M_d(f)(x) > 2\lambda\}\big|\,d\lambda$$

and we use Theorem 3.4.4 to obtain the following sequence of inequalities:

$$\begin{aligned} I_N \ \le\ & 2^p\int_0^{\frac{N}{2}} p\lambda^{p-1}\big|\{x \in \mathbf{R}^n:\ M_d(f)(x) > 2\lambda,\ M^\#(f)(x) \le \gamma\lambda\}\big|\,d\lambda \\ & + 2^p\int_0^{\frac{N}{2}} p\lambda^{p-1}\big|\{x \in \mathbf{R}^n:\ M^\#(f)(x) > \gamma\lambda\}\big|\,d\lambda \end{aligned}$$