(2) For all cubes Q in $\mathbf{R}^n$ we have

$$\frac{1}{2}M^{\#}(f)(x) \le \sup_{Q \ni x} \inf_{a \in \mathbf{C}} \frac{1}{|Q|}\int_Q |f(y)-a|\,dy \le M^{\#}(f)(x).$$

(3) $M^{\#}(|f|) \le 2M^{\#}(f)$.

(4) We have $M^{\#}(f+g) \le M^{\#}(f) + M^{\#}(g)$.

Proof. The proof of (1) is trivial. To prove (2) we fix $\varepsilon > 0$ and for any cube Q we pick a constant a_Q such that

$$\frac{1}{|Q|}\int_Q |f(y)-a_Q|\,dy \le \inf_{a \in Q} \frac{1}{|Q|}\int_Q |f(y)-a|\,dy + \varepsilon\,.$$

Then

$$\begin{aligned}
\frac{1}{|Q|}\int_Q \big|f(y) - \operatorname*{Avg}_Q f\big|\,dy \;&\le\; \frac{1}{|Q|}\int_Q |f(y)-a_Q|\,dy + \frac{1}{|Q|}\int_Q \big|\operatorname*{Avg}_Q f - a_Q\big|\,dy \\
&\le\; \frac{1}{|Q|}\int_Q |f(y)-a_Q|\,dy + \frac{1}{|Q|}\int_Q |f(y)-a_Q|\,dy \\
&\le\; 2\inf_{a \in Q} \frac{1}{|Q|}\int_Q |f(y)-a|\,dy + 2\varepsilon\,.
\end{aligned}$$

Taking the supremum over all cubes Q in $\mathbf{R}^n$, we obtain the first inequality in (2), since $\varepsilon > 0$ was arbitrary. The other inequality in (2) is simple. The proofs of (3) and (4) are immediate. □

We saw that $M^{\#}(f) \le 2M_c(f)$, which implies that

$$\big\|M^{\#}(f)\big\|_{L^p} \le C_n p(p-1)^{-1}\big\|f\big\|_{L^p} \tag{3.4.1}$$

for $1 < p < \infty$. Thus the sharp function of an L^p function is also in L^p whenever $1 < p < \infty$. The fact that the converse inequality is also valid is one of the main results in this section. We obtain this estimate via a distributional inequality for the sharp function called a *good lambda* inequality.

3.4.2 A Good Lambda Estimate for the Sharp Function

A useful tool in obtaining the converse inequality to (3.4.1) is the dyadic maximal function.

Definition 3.4.3. A dyadic cube is a set of the form $\prod_{j=1}^{m}[m_j 2^{-k}, (m_j+1)2^{-k})$, where $m_1, \dots, m_n, k \in \mathbf{Z}$. Given a locally integrable function f on $\mathbf{R}^n$, we define its *dyadic maximal function* $M_d(f)$ by