3.3.2. Let $x_0 \in \mathbf{R}^n$ and $\mu = \delta_{(x_0,1)}$ be the Dirac mass at the point $(x_0, 1)$. Show that μ is Carleson measure and compute $\|\mu\|_{\mathscr{C}}^{\text{cylinder}}$ and $\|\mu\|_{\mathscr{C}}$. Which of these norms is larger?

3.3.3. Define *conical* and *hemispherical* tents over balls in $\mathbf{R}^n$ as well as *pyramidal* tents over cubes in $\mathbf{R}^n$ and define the expressions $\|\mu\|_{\mathscr{C}}^{\text{cone}}$, $\|\mu\|_{\mathscr{C}}^{\text{hemisphere}}$, and $\|\mu\|_{\mathscr{C}}^{\text{pyramid}}$. Show that

$$\|\mu\|_{\mathscr{C}}^{\text{cone}} \approx \|\mu\|_{\mathscr{C}}^{\text{hemisphere}} \approx \|\mu\|_{\mathscr{C}}^{\text{pyramid}} \approx \|\mu\|_{\mathscr{C}},$$

where all the implicit constants in the previous estimates depend only on the dimension.

3.3.4. Suppose that Φ has a radial, bounded, symmetrically decreasing integrable majorant. Set $F(x,t) = (f * \Phi_t)(x)$, where f is a locally integrable function on $\mathbf{R}^n$. Prove that

$$F^*(x) \le C_{\Phi} M(f)(x),$$

where M is the Hardy–Littlewood maximal operator and C_Φ is a constant that depends only on the decreasing majorant of Φ.
$\big[$*Hint:* If $\varphi(|x|)$ is the claimed majorant of $\Phi(x)$, then the function $\psi(|x|) = \varphi(0)$ for $|x| \le 1$ and $\psi(|x|) = \varphi(|x| - 1)$ for $|x| \ge 1$ is a majorant for the function $\Psi(x) = \sup_{|u|\le 1} |\Phi(x-u)|$.$\big]$

3.3.5. Let F be a function on $\mathbf{R}^{n+1}_+$, let F^* be the nontangential maximal function derived from F, and let $\mu \ge 0$ be a measure on $\mathbf{R}^{n+1}_+$. Prove that

$$\big\|F\big\|_{L^r(\mathbf{R}^{n+1}_+,\mu)} \le C_n^{1/r} \left(\int_{\mathbf{R}^n} \mathscr{C}(\mu)(x) F^*(x)^r \, dx \right)^{1/r},$$

where C_n is the constant of Theorem 3.3.5 and $0 < r < \infty$.

3.3.6. (a) Given A a closed subset of $\mathbf{R}^n$ and $0 < \gamma < 1$, define

$$A^*_\gamma = \left\{ x \in \mathbf{R}^n : \inf_{r>0} \frac{|A \cap B(x,r)|}{|B(x,r)|} \ge \gamma \right\}.$$

Show that $A_\gamma{}^*$ is a closed subset of A and that it satisfies

$$|(A^*_\gamma)^c| \le \frac{3^n}{1-\gamma} |A^c|.$$

$\big[$*Hint:* Consider the Hardy–Littlewood maximal function of χ_{A^c}.$\big]$