and let $\delta_{2^{-j}}(t)$ be Dirac mass at the point $t = 2^{-j}$. Then there is a constant $C_{n,\delta}$ such that

$$d\mu(x,t) = \sum_{j\in\mathbf{Z}} |(\Psi_{2^{-j}} * b)(x)|^2 \, dx \, \delta_{2^{-j}}(t)$$

is a Carleson measure on $\mathbf{R}^{n+1}_+$ with norm at most $C_{n,\delta}(A+B)^2\|b\|^2_{BMO}$.
(b) Suppose that

$$\sup_{\xi\in\mathbf{R}^n} \int_0^\infty |\widehat{\Psi}(t\xi)|^2 \frac{dt}{t} \le B^2 < \infty. \tag{3.3.14}$$

Then the continuous version $d\nu(x,t)$ of $d\mu(x,t)$ defined by

$$d\nu(x,t) = |(\Psi_t * b)(x)|^2 \, dx \frac{dt}{t}$$

is a Carleson measure on $\mathbf{R}^{n+1}_+$ with norm at most $C_{n,\delta}(A+B)^2\|b\|^2_{BMO}$ for some constant $C_{n,\delta}$.
(c) Let $\delta, A > 0$. Suppose that $\{K_t\}_{t>0}$ are functions on $\mathbf{R}^n \times \mathbf{R}^n$ that satisfy

$$|K_t(x,y)| \le \frac{A\, t^\delta}{(t+|x-y|)^{n+\delta}} \tag{3.3.15}$$

for all $t > 0$ and all $x, y \in \mathbf{R}^n$. Let R_t be the linear operator

$$R_t(f)(x) = \int_{\mathbf{R}^n} K_t(x,y) f(y)\, dy,$$

which is well defined for all $f \in \bigcup_{1\le p\le\infty} L^p(\mathbf{R}^n)$. Suppose that $R_t(1) = 0$ for all $t > 0$ and that there is a constant $B > 0$ such that

$$\int_0^\infty \int_{\mathbf{R}^n} \big|R_t(f)(x)\big|^2 \frac{dx\,dt}{t} \le B^2 \big\|f\big\|^2_{L^2(\mathbf{R}^n)} \tag{3.3.16}$$

for all $f \in L^2(\mathbf{R}^n)$. Then for all b in BMO the measure

$$\big|R_t(b)(x)\big|^2 \frac{dx\,dt}{t}$$

is Carleson with norm at most a constant multiple of $(A+B)^2\|b\|^2_{BMO}$.

We note that if, in addition to (3.3.12), the function Ψ has mean value zero and satisfies $|\nabla\Psi(x)| \le A(1+|x|)^{-n-\delta}$, then (3.3.13) and (3.3.14) hold and therefore conclusions (a) and (b) of Theorem 3.3.8 follow. (See for instance [156, Page 422]).

Proof. We prove (a). The measure μ is defined so that for every μ-integrable function F on $\mathbf{R}^{n+1}_+$ we have

$$\int_{\mathbf{R}^{n+1}_+} F(x,t)\, d\mu(x,t) = \sum_{j\in\mathbf{Z}} \int_{\mathbf{R}^n} |(\Psi_{2^{-j}} * b)(x)|^2 F(x, 2^{-j})\, dx. \tag{3.3.17}$$

For a cube Q in $\mathbf{R}^n$ we let Q^* be the cube with the same center and orientation whose side length is $3\sqrt{n}\,\ell(Q)$, where $\ell(Q)$ is the side length of Q. Fix a cube Q in $\mathbf{R}^n$, take F to be the characteristic function of the tent of Q, and split b as

$$b = \big(b - \operatorname*{Avg}_Q b\big)\chi_{Q^*} + \big(b - \operatorname*{Avg}_Q b\big)\chi_{(Q^*)^c} + \operatorname*{Avg}_Q b.$$

Since Ψ has mean value zero, $\Psi_{2^{-j}} * \operatorname{Avg}_Q b = 0$. Then (3.3.17) gives

$$\mu(T(Q)) = \sum_{2^{-j} \le \ell(Q)} \int_Q |\Delta_j(b)(x)|^2\,dx \le 2\Sigma_1 + 2\Sigma_2,$$

where

$$\begin{aligned}
\Sigma_1 &= \sum_{j\in\mathbf{Z}} \int_{\mathbf{R}^n} \big|\Delta_j\big((b - \operatorname*{Avg}_Q b)\chi_{Q^*}\big)(x)\big|^2\,dx, \\
\Sigma_2 &= \sum_{2^{-j}\le \ell(Q)} \int_Q \big|\Delta_j\big((b - \operatorname*{Avg}_Q b)\chi_{(Q^*)^c}\big)(x)\big|^2\,dx.
\end{aligned}$$

Using Plancherel's theorem and (3.3.13), we obtain

$$\begin{aligned}
\Sigma_1 &\le \sup_{\xi} \sum_{j\in\mathbf{Z}} |\widehat{\Psi}(2^{-j}\xi)|^2 \int_{\mathbf{R}^n} \big|\big((b - \operatorname*{Avg}_Q b)\chi_{Q^*}\big)\widehat{\ }(\eta)\big|^2\,d\eta \\
&\le B^2 \int_{Q^*} \big|b(x) - \operatorname*{Avg}_Q b\big|^2\,dx \\
&\le 2B^2 \int_{Q^*} \big|b(x) - \operatorname*{Avg}_{Q*} b\big|^2\,dx + 2B^2|Q^*|\,\big|\operatorname*{Avg}_{Q*} b - \operatorname*{Avg}_Q b\big|^2 \\
&\le B^2 \int_{Q^*} \big|b(x) - \operatorname*{Avg}_{Q^*} b\big|^2\,dx + c_n 2B^2 \|b\|_{BMO}^2\,|Q| \\
&\le C_n B^2 \|b\|_{BMO}^2\,|Q|,
\end{aligned}$$

where the used the analogue of (3.1.4) for cubes and Corollary 3.1.8. To estimate Σ_2, we use the size estimate of the function Ψ. We obtain

$$\big|\big(\Psi_{2^{-j}} * \big(b - \operatorname*{Avg}_Q b\big)\chi_{(Q^*)^c}\big)(x)\big| \le \int_{(Q^*)^c} \frac{A\,2^{-j\delta}\,|b(y) - \operatorname{Avg}_Q b|}{(2^{-j} + |x-y|)^{n+\delta}}\,dy. \tag{3.3.18}$$

But note that if c_Q is the center of Q, then

$$\begin{aligned}
2^{-j} + |x-y| &\ge |y-x| \\
&\ge |y - c_Q| - |c_Q - x| \\
&\ge \frac{1}{2}|c_Q - y| + \frac{3\sqrt{n}}{4}\ell(Q) - |c_Q - x|
\end{aligned}$$