$$\begin{aligned}\sup_Q \frac{1}{|Q|}\int_Q |b(x)-C_Q|\,dx &= \sup_Q \frac{1}{|Q|}\int_Q |F^Q(x)|\,dx\\ &\le \sup_Q |Q|^{-1}|Q|^{\frac12}\big\|F^Q\big\|_{L^2(Q)}\\ &\le \sup_Q |Q|^{-\frac12}\big\|L\big\|_{L^2_0(Q)\to\mathbf{C}}\\ &\le c_n\big\|L\big\|_{H^1\to\mathbf{C}}<\infty .\end{aligned}$$

Using Proposition 3.1.2 (3), we deduce that $b\in BMO$ and $\|b\|_{BMO}\le 2c_n\|L\|_{H^1\to\mathbf{C}}$. Finally, (3.2.11) implies that

$$L(g)=\int_{\mathbf{R}^n} b(x)g(x)\,dx = L_b(g)$$

for all $g\in H^1_0(\mathbf{R}^n)$, proving that the linear functional $L$ coincides with $L_b$ on a dense subspace of $H^1$. Consequently, $L=L_b$, and this concludes the proof of part (b). □

## Exercises

**3.2.1.** Given $b$ in $BMO$, let $L_b$ be as in Definition 3.2.1. Prove that for $b$ in $BMO$ we have

$$\big\|b\big\|_{BMO} \approx \sup_{\|f\|_{H^1}\le 1}\big|L_b(f)\big|,$$

and for a given $f$ in $H^1$ we have

$$\big\|f\big\|_{H^1} \approx \sup_{\|b\|_{BMO}\le 1}\big|L_b(f)\big|.$$

$\big[$*Hint:* Use $\|T\|_{X^*}=\sup_{\substack{x\in X\\ \|x\|_X\le 1}}|T(x)|$ for all $T$ in the dual of a Banach space $X$.$\big]$

**3.2.2.** Suppose that a locally integrable function $u$ is supported in a cube $Q$ in $\mathbf{R}^n$ and satisfies

$$\int_Q u(x)g(x)\,dx=0$$

for all ~~square integrable~~ bounded functions $g$ on $Q$ with mean value zero. Show that $u$ is almost everywhere equal to a constant.

## 3.3 Nontangential Maximal Functions and Carleson Measures

Many properties of functions defined on $\mathbf{R}^n$ are related to corresponding properties of associated functions defined on $\mathbf{R}^{n+1}_+$ in a natural way. A typical example of this situation is the relation between an $L^p(\mathbf{R}^n)$ function $f$ and its Poisson integral

$f * P_t$ or more generally $f * \Phi_t$, where $\{\Phi_t\}_{t>0}$ is an approximate identity. Here $\Phi$ is a Schwartz function on $\mathbf{R}^n$ with integral 1. A maximal operator associated to the approximate identity $\{f * \Phi_t\}_{t>0}$ is

$$f \to \sup_{t>0} |f * \Phi_t|,$$

which we know is pointwise controlled by a multiple of the Hardy–Littlewood maximal function $M(f)$. Another example of a maximal operator associated to the previous approximate identity is the nontangential maximal function

$$f \to M^*(f;\Phi)(x) = \sup_{t>0} \sup_{|y-x|<t} |(f * \Phi_t)(y)|.$$

To study nontangential behavior we consider general functions $F$ defined on $\mathbf{R}^{n+1}_+$ that are not necessarily given as an average of functions defined on $\mathbf{R}^n$. Throughout this section we use capital letters to denote functions defined on $\mathbf{R}^{n+1}_+$. When we write $F(x,t)$ we mean that $x \in \mathbf{R}^n$ and $t > 0$.

## *3.3.1 Definition and Basic Properties of Carleson Measures*

**Definition 3.3.1.** Let $F$ be a measurable function on $\mathbf{R}^{n+1}_+$. For $x$ in $\mathbf{R}^n$ let $\Gamma(x)$ be the cone with vertex $x$ defined by

$$\Gamma(x) = \{(y,t) \in \mathbf{R}^n \times \mathbf{R}^+ :\ |y-x| < t\}.$$

A picture of this cone is shown in Figure 3.1. The *nontangential maximal function* of $F$ is the function

$$F^*(x) = \sup_{(y,t)\in\Gamma(x)} |F(y,t)|$$

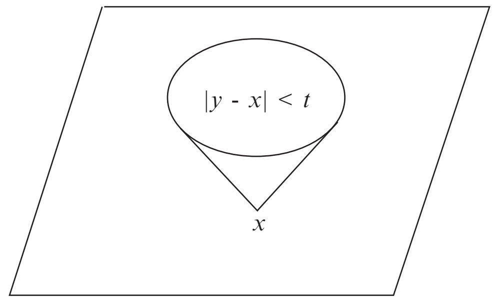
defined on $\mathbf{R}^n$. This function is obtained by taking the supremum of the values of $F$ inside the cone $\Gamma(x)$.

We observe that if $F^*(x) = 0$ for almost all $x \in \mathbf{R}^n$, then $F$ is identically equal to zero on $\mathbf{R}^{n+1}_+$. To establish this claim, suppose that $|F(x_0,t_0)| > 0$ for some point $(x_0,t_0) \in \mathbf{R}^n \times \mathbf{R}^+$. Then for all $z$ with $|z-x_0| < t_0$ we have $(x_0,t_0) \in \Gamma(z)$, hence $F^*(z) \ge |F(x_0,t_0)| > 0$. Thus $F^* > 0$ on the ball $B(x_0,t_0)$, which is a set of positive measure, a contradiction.

**Definition 3.3.2.** Given a ball $B = B(x_0,r)$ in $\mathbf{R}^n$ we define the *cylindrical tent* over $B$ to be the "cylindrical set"

$$T(B) = \{(x,t) \in \mathbf{R}^{n+1}_+ :\ x \in B, \quad 0 < t \le r\}.$$

**Fig. 3.1** The cone $\Gamma(x)$ truncated at height $t$.




For a cube $Q$ in $\mathbf{R}^n$ we define the *tent* over $Q$ to be the cube

$$T(Q) = Q \times (0, \ell(Q)] .$$

A tent over a ball and over a cube are shown in Figure 3.2. A positive Borel measure $\mu$ on $\mathbf{R}^{n+1}_+$ is called a *Carleson measure* if

$$\big\|\mu\big\|_{\mathscr{C}} = \sup_Q \frac{1}{|Q|}\mu(T(Q)) < \infty, \tag{3.3.1}$$

where the supremum in (3.3.1) is taken over all cubes $Q$ in $\mathbf{R}^n$. The *Carleson function* of the measure $\mu$ is defined as

$$\mathscr{C}(\mu)(x) = \sup_{Q \ni x} \frac{1}{|Q|}\mu(T(Q)), \tag{3.3.2}$$

where the supremum in (3.3.2) is taken over all cubes in $\mathbf{R}^n$ containing the point $x$. Observe that $\|\mathscr{C}(\mu)\|_{L^\infty} = \|\mu\|_{\mathscr{C}}$.

We also define

$$\big\|\mu\big\|^{\text{cylinder}}_{\mathscr{C}} = \sup_B \frac{1}{|B|}\mu(T(B)), \tag{3.3.3}$$

where the supremum is taken over all balls $B$ in $\mathbf{R}^n$. One can easily verify that there exist dimensional constants $c_n$ and $C_n$ such that

$$c_n\big\|\mu\big\|_{\mathscr{C}} \le \big\|\mu\big\|^{\text{cylinder}}_{\mathscr{C}} \le C_n\big\|\mu\big\|_{\mathscr{C}}$$

for all Borel measures $\mu$ on $\mathbf{R}^{n+1}_+$, that is, a measure satisfies the Carleson condition (3.3.1) with respect to cubes if and only if it satisfies the analogous condition (3.3.3) with respect to balls. Likewise, the Carleson function $\mathscr{C}(\mu)$ defined with respect to tents over cubes is comparable to

$$\mathscr{C}^{\text{cylinder}}(\mu)(x) = \sup_{B \ni x} \frac{1}{|B|}\mu(T(B)),$$

defined with respect to cylindrical tents over balls $B$ in $\mathbf{R}^n$.

**Examples 3.3.3.** The Lebesgue measure on $\mathbf{R}_+^{n+1}$ is not a Carleson measure. Indeed, it is not difficult to see that condition (3.3.1) cannot hold for large balls.

Let $L$ be a line in $\mathbf{R}^2$. For $A$ measurable subsets of $\mathbf{R}_+^2$ define $\mu(A)$ to be the linear Lebesgue measure of the set $L \cap A$. Then $\mu$ is a Carleson measure on $\mathbf{R}_+^2$. Indeed, the linear measure of the part of a line inside the box $[x_0 - r, x_0 + r] \times (0, r]$ is at most equal to the diagonal of the box, that is, $\sqrt{5}r$.

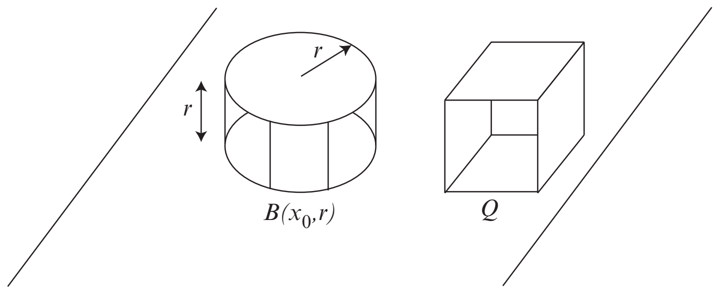



**Fig. 3.2** The tents over the ball $B(x_0, r)$ and over a cube $Q$ in $\mathbf{R}^2$.

Likewise, let $P$ be an affine plane in $\mathbf{R}^{n+1}$ and define a measure $\nu$ by setting $\nu(A)$ to be the $n$-dimensional Lebesgue measure of the set $A \cap P$ for any $A \subseteq \mathbf{R}_+^{n+1}$. A similar idea shows that $\nu$ is a Carleson measure on $\mathbf{R}_+^{n+1}$.

We now turn to the study of some interesting boundedness properties of functions on $\mathbf{R}_+^{n+1}$ with respect to Carleson measures.

A useful tool in this study is the *Whitney decomposition* of an open set in $\mathbf{R}^n$. This is a decomposition of a general open set $\Omega$ in $\mathbf{R}^n$ as a union of disjoint cubes whose lengths are proportional to their distance from the boundary of the open set. For a given cube $Q$ in $\mathbf{R}^n$, we denote by $\ell(Q)$ its length.

**Proposition 3.3.4.** *(**Whitney decomposition**) Let $\Omega$ be an open nonempty proper subset of $\mathbf{R}^n$. Then there exists a family of closed cubes $\{Q_j\}_j$ such that*
*(a) $\bigcup_j Q_j = \Omega$ and the $Q_j$'s have disjoint interiors;*
*(b) $\sqrt{n}\,\ell(Q_j) \le dist\,(Q_j, \Omega^c) \le 4\sqrt{n}\,\ell(Q_j)$;*
*(c) if the boundaries of two cubes $Q_j$ and $Q_k$ touch, then*

$$\frac{1}{4} \le \frac{\ell(Q_j)}{\ell(Q_k)} \le 4\,;$$

*(d) for a given $Q_j$ there exist at most $12^n$ $Q_k$'s that touch it.*

The proof of Proposition 3.3.4 is given in Appendix J in [156].

**Theorem 3.3.5.** *There exists a dimensional constant $C_n$ such that for all $\alpha > 0$, all Borel measures $\mu \ge 0$ on $\mathbf{R}_+^{n+1}$, and all $\mu$-measurable functions $F$ on $\mathbf{R}_+^{n+1}$, the set $\Omega_\alpha = \{F^* > \alpha\}$ is open (thus Lebesgue measurable) and we have*

$$\mu\big(\{(x,t) \in \mathbf{R}_+^{n+1} :\ |F(x,t)| > \alpha\}\big) \le C_n \int_{\{F^* > \alpha\}} \mathscr{C}(\mu)(x)\,dx. \tag{3.3.4}$$