$$\|g\|_{H^1} \le c_n |Q|^{\frac{1}{2}} \|g\|_{L^2}. \tag{3.2.4}$$

To prove (3.2.4) we use the square function characterization of H^1. We fix a Schwartz function Ψ on $\mathbf{R}^n$ whose Fourier transform is supported in the annulus $\frac{1}{2} \le |\xi| \le 2$ and that satisfies (1.3.6) for all $\xi \ne 0$ and we let $\Delta_j(g) = \Psi_{2^{-j}} * g$. To estimate the L^1 norm of $\big(\sum_j |\Delta_j(g)|^2\big)^{1/2}$ over $\mathbf{R}^n$, consider the part of the integral over $3\sqrt{n}Q$ and the integral over $(3\sqrt{n}Q)^c$. First we use Hölder's inequality and an L^2 estimate to prove that

$$\int_{3\sqrt{n}Q} \Big(\sum_j |\Delta_j(g)(x)|^2\Big)^{\frac{1}{2}} dx \le c_n |Q|^{\frac{1}{2}} \|g\|_{L^2}.$$

Now for $x \notin 3\sqrt{n}Q$ we use the mean value property of g to obtain

$$|\Delta_j(g)(x)| \le \frac{c_n \|g\|_{L^2} 2^{nj+j} |Q|^{\frac{1}{n}+\frac{1}{2}}}{(1+2^j|x-c_Q|)^{n+2}}, \tag{3.2.5}$$

where c_Q is the center of Q. Estimate (3.2.5) is obtained in a way similar to that we obtained the corresponding estimate for one atom; see Theorem 2.3.11 for details. Now (3.2.5) implies that

$$\int_{(3\sqrt{n}Q)^c} \Big(\sum_j |\Delta_j(g)(x)|^2\Big)^{\frac{1}{2}} dx \le c_n |Q|^{\frac{1}{2}} \|g\|_{L^2},$$

which proves (3.2.4).

Since $L_0^2(Q)$ is a subspace of H^1, it follows from (3.2.4) that the linear functional $L : H^1 \to \mathbf{C}$ is also a bounded linear functional on $L_0^2(Q)$ with norm

$$\|L\|_{L_0^2(Q)\to\mathbf{C}} \le c_n |Q|^{1/2} \|L\|_{H^1\to\mathbf{C}}. \tag{3.2.6}$$

By the Riesz representation theorem for the Hilbert space $L_0^2(Q)$, there is an element F^Q in $(L_0^2(Q))^* = L^2(Q)/\{\text{constants}\}$, equipped with norm $\|h\| = \inf_{c\in\mathbf{C}} \|h-c\|_{L^2(Q)}$, such that

$$L(g) = \int_Q F^Q(x) g(x)\, dx, \tag{3.2.7}$$

for all $g \in L_0^2(Q)$, and this F^Q satisfies

$$\|F^Q\|_{L^2(Q)} \le \|L\|_{L_0^2(Q)\to\mathbf{C}}. \tag{3.2.8}$$

Thus for any cube Q in $\mathbf{R}^n$, there is square integrable function F^Q supported in Q such that (3.2.7) is satisfied. We observe that if a cube Q is contained in another cube Q', then F^Q differs from $F^{Q'}$ by a constant on Q. Indeed, for all $g \in L_0^2(Q)$ we have

$$\int_Q F^{Q'}(x) g(x)\, dx = L(g) = \int_Q F^Q(x) g(x)\, dx$$