integrable over any cube. Thus given a BMO function b on $\mathbf{R}^n$ and an L^2 function g with integral zero supported in a cube Q in $\mathbf{R}^n$, the integral $\int_{\mathbf{R}^n} g(x)\,b(x)\,dx$ converges absolutely by the Cauchy–Schwarz inequality.

Definition 3.2.1. Denote by $H_0^1(\mathbf{R}^n)$ the space of all finite linear combinations of L^2 atoms for $H^1(\mathbf{R}^n)$ and fix $b \in BMO(\mathbf{R}^n)$. Given $g \in H_0^1$ we define a linear functional

$$L_b(g) = \int_{\mathbf{R}^n} g(x)b(x)\,dx \tag{3.2.1}$$

as an absolutely convergent integral. Observe that the integral in (3.2.1) and thus the definition of L_b on H_0^1 remain the same if b is replaced by $b+c$, where c is an additive constant. Additionally, we observe that (3.2.1) is also an absolutely convergent integral when g is a general element of $H^1(\mathbf{R}^n)$ and the BMO function b is bounded.

To extend the definition of L_b on the entire H^1 for all functions b in BMO we need to know that

$$\big\|L_b\big\|_{H^1\to\mathbf{C}} \le C_n \|b\|_{BMO}\,, \qquad \text{whenever } b \text{ is bounded,} \tag{3.2.2}$$

a fact that will be proved momentarily. Assuming (3.2.2), take $b \in BMO$ and let $b_M(x) = b\chi_{|b|\le M}$ for $M = 1,2,3,\dots$. Since $\|b_M\|_{BMO} \le \frac{9}{4}\|b\|_{BMO}$ (Exercise 3.1.4), the sequence of linear functionals $\{L_{b_M}\}_M$ lies in a multiple of the unit ball of $(H^1)^*$ and by the Banach–Alaoglou theorem there is a subsequence $M_j \to \infty$ as $j \to \infty$ such that $L_{b_{M_j}}$ converges weakly to a bounded linear functional $\widetilde{L}_b$ on H^1. This means that for all f in $H^1(\mathbf{R}^n)$ we have

$$L_{b_{M_j}}(f) \to \widetilde{L}_b(f)$$

as $j \to \infty$.

If a_Q is a fixed L^2 atom for H^1, the difference $|L_{b_{M_j}}(a_Q) - L_b(a_Q)|$ is bounded by $\|a_Q\|_{L^2}\ (\|b_{M_j} - \mathrm{Avg}_Q\, b_{M_j} - b + \mathrm{Avg}_Q\, b\|_{L^2(Q)})$ which is in turn bounded by $\|a_Q\|_{L^2}(\|b_{M_j} - b\|_{L^2(Q)} + |Q|^{1/2}|\,\mathrm{Avg}_Q(b_{M_j} - b)|)$, and this expression tends to zero as $j \to \infty$ by the Lebesgue dominated convergence theorem. The same conclusion holds for any finite linear combination of a_Q's. Thus for all $g \in H_0^1$ we have

$$L_{b_{M_j}}(g) \to L_b(g)\,,$$

and consequently, $L_b(g) = \widetilde{L}_b(g)$ for all $g \in H_0^1$. Since H_0^1 is dense in H^1 and L_b and $\widetilde{L}_b$ coincide on H_0^1, it follows that $\widetilde{L}_b$ is the *unique* bounded extension of L_b on H^1. We have therefore defined L_b on the entire H^1 as a weak limit of bounded linear functionals.

Having set the definition of L_b, we proceed by showing the validity of (3.2.2). Let b be a bounded BMO function. Given f in H^1, find a sequence a_k of L^2 atoms for H^1 supported in cubes Q_k such that

$$f = \sum_{k=1}^{\infty} \lambda_k a_k \tag{3.2.3}$$