Show that f satisfies the estimate

$$\left|\left\{x \in Q : \left|f(x) - \operatorname*{Avg}_Q f\right| > \alpha\right\}\right| \le |Q|\, e^{-c\alpha^{1/m}}$$

with $c = (2b)^{-1/m} \log 2$.
[*Hint:* Try $p = (\alpha/2b)^{1/m}$.]

3.1.8. Prove that $\big|\log|x|\big|^p$ is not in $BMO(\mathbf{R})$ when $1 < p < \infty$.
[*Hint:* Show that if $\big|\log|x|\big|^p$ were in BMO, then estimate (3.1.9) would be violated for large α.]

3.1.9. Given $1 < p < \infty$ and f locally integrable on $\mathbf{R}^n$ prove that

$$\sup_Q \left(\inf_{c_Q} \frac{1}{|Q|} \int_Q |f(x) - c_Q|^p \, dx\right)^{\frac{1}{p}} \approx \|f\|_{BMO}.$$

[*Hint:* Use Proposition 3.1.2 (4) and Corollary 3.1.9.]

3.1.10. Let $f \in BMO(\mathbf{R})$ have mean value equal to zero on a fixed closed interval I. Find a BMO function g on $\mathbf{R}$ such that

(1) $g = f$ on I;

(2) $g = 0$ on $\mathbf{R} \setminus \frac{5}{3}I$;

(3) $\|g\|_{BMO} \le 12\, \|f\|_{BMO}$.

[*Hint:* Let I_0 be the closed middle third of I. Write the interior of I as $\bigcup_{k\in\mathbf{Z}} I_k$, where for $|k| > 0$, I_k are closed subintervals of I such that the right endpoint of I_k coincides with the left endpoint of I_{k+1} and $\operatorname{dist}(I_k, \partial I) = |I_k| = \frac{1}{3} 2^{-|k|}$. For $|k| \ge 1$, let J_k be the reflection of I_k with respect to the closest endpoint of I and set $g = \operatorname{Avg}_{I_k} f$ on J_k for $|k| > 1$, $g = f$ on I, and zero otherwise. To prove property (3), given an arbitrary interval Q on the real line, consider the cases where $|Q| \ge \frac{1}{3}|I|$ and $|Q| < \frac{1}{3}|I|$.]

3.2 Duality between H^1 and BMO

The next result we discuss is a remarkable duality relationship between the Hardy space H^1 and BMO. Precisely, we show that BMO is the dual space of H^1. This means that every continuous linear functional on the Hardy space H^1 can be realized as integration against a fixed BMO function, where *integration* in this context is an abstract operation, not necessarily given by an absolutely convergent integral. Restricting our attention, however, to a dense subspace of H^1 such as the space of all finite sums of atoms, the use of the word *integration* is well justified. Indeed, first we note that an important consequence of (3.1.15) is that any BMO function b lies in $L^p(Q)$ for any Q in $\mathbf{R}^n$ and any p satisfying $1 < p < \infty$; in particular it is square