We apply the Calderón–Zygmund decomposition to the function $f - \operatorname{Avg}_Q f$ inside the cube Q. We introduce the following selection criterion for a cube R:

$$\frac{1}{|R|}\int_R \big|f(x) - \operatorname*{Avg}_{Q} f\big|\,dx > b. \tag{3.1.10}$$

Since

$$\frac{1}{|Q|}\int_Q \big|f(x) - \operatorname*{Avg}_{Q} f\big|\,dx \le \big\|f\big\|_{BMO} = 1 < b\,,$$

the cube Q does not satisfy the selection criterion (3.1.10). Set $Q^{(0)} = Q$ and subdivide $Q^{(0)}$ into 2^n equal closed subcubes of side length equal to half of the side length of Q. Select such a subcube R if it satisfies the selection criterion (3.1.10). Now subdivide all nonselected cubes into 2^n equal subcubes of half their side length by bisecting the sides, and select among these subcubes those that satisfy (3.1.10). Continue this process indefinitely. We obtain a countable collection of cubes $\{Q_j^{(1)}\}_j$ satisfying the following properties:

(A-1) The interior of every $Q_j^{(1)}$ is contained in $Q^{(0)}$.

(B-1) $b < |Q_j^{(1)}|^{-1}\displaystyle\int_{Q_j^{(1)}} \big|f(x) - \operatorname*{Avg}_{Q^{(0)}} f\big|\,dx \le 2^n b.$

(C-1) $\Big|\operatorname*{Avg}_{Q_j^{(1)}} f - \operatorname*{Avg}_{Q^{(0)}} f\Big| \le 2^n b.$

(D-1) $\displaystyle\sum_j |Q_j^{(1)}| \le \frac{1}{b}\sum_j \int_{Q_j^{(1)}} \big|f(x) - \operatorname*{Avg}_{Q^{(0)}} f\big|\,dx \le \frac{1}{b}|Q^{(0)}|.$

(E-1) $\big|f - \operatorname*{Avg}_{Q^{(0)}} f\big| \le b$ a.e. on the set $Q^{(0)} \setminus \bigcup_j Q_j^{(1)}$.

We call the cubes $Q_j^{(1)}$ of first generation. Note that the first inequality in (D-1) requires (B-1) while the second requires the fact that $Q^{(0)}$ does not satisfy (3.1.10).

We now fix a selected first-generation cube $Q_j^{(1)}$ and we introduce the following selection criterion for a cube R:

$$\frac{1}{|R|}\int_R \big|f(x) - \operatorname*{Avg}_{Q_j^{(1)}} f\big|\,dx > b. \tag{3.1.11}$$

Observe that $Q_j^{(1)}$ does not satisfy the selection criterion (3.1.11). We apply a similar Calderón–Zygmund decomposition to the function

$$f - \operatorname*{Avg}_{Q_j^{(1)}} f$$