Let $G_{|x|}(s)$ be the function inside the square brackets in (2.4.56). Then $G_{|x|}(s) \to 0$ as $|x| \to \infty$ for all s. The hypothesis (2.4.42) implies that $G_{|x|}$ is bounded by a constant and it is therefore integrable over the interval $\big[\frac{1}{2}\varepsilon, \infty\big)$ with respect to the measure $s^{-n-1}ds$. By the Lebesgue dominated convergence theorem we deduce that the expression in (2.4.56) converges to zero as $|x| \to \infty$ and thus it can be made smaller that $\eta/2$ for $|x| \geq R_2$, for some constant R_2. Then with $R_0'' = \sqrt{2}R_2$ we have that if $|(x,t)| \geq R_0''$ then (2.4.56) is at most $\eta/2$. Since $U - V \leq U$, we deduce the validity of (2.4.54) for $|(x,t)| > R_0 = \max(R_0', R_0'')$.

Let $r = p/q > 1$. Assumption (2.4.42) implies that the functions $x \mapsto |F(x,t)|^q$ are in L^r uniformly in t. Since any closed ball of L^r is weak* compact, there is a sequence $\varepsilon_k \to 0$ such that $|F(x,\varepsilon_k)|^q \to h$ weakly in L^r as $k \to \infty$ to some function $h \in L^r$. Since $P_t \in L^{r'}$, this implies that

$$(|F(\cdot,\varepsilon_k)|^q * P_t)(x) \to (h * P_t)(x)$$

for all $x \in \mathbf{R}^n$. Using (2.4.53) we obtain

$$|F(x,t)|^q = \limsup_{k\to\infty} |F(x,t+\varepsilon_k)|^q \leq \limsup_{k\to\infty} \big(|F(\,\cdot\,,\varepsilon_k)|^q * P_t\big)(x) = (h * P_t)(x)\,,$$

which gives for all $x \in \mathbf{R}^n$,

$$|F|^*(x) \leq \Big[\sup_{t>0} \sup_{|y-x|<t} (|h| * P_t)(y)\Big]^{1/q} \leq C_n' M(h)(x)^{1/q}\,. \tag{2.4.57}$$

Let $g \in L^{r'}(\mathbf{R}^n)$ with $L^{r'}$ norm at most one. The weak convergence yields

$$\int_{\mathbf{R}^n} |F(x,\varepsilon_k)|^q g(x)\,dx \to \int_{\mathbf{R}^n} h(x)\,g(x)\,dx$$

as $k \to \infty$, and consequently we have

$$\left|\int_{\mathbf{R}^n} h(x)\,g(x)\,dx\right| \leq \sup_k \int_{\mathbf{R}^n} |F(x,\varepsilon_k)|^q |g(x)|\,dx \leq \big\|g\big\|_{L^{r'}} \sup_{t>0} \bigg(\int_{\mathbf{R}^n} |F(x,t)|^p\,dx\bigg)^{\frac{1}{r}}.$$

Since g is arbitrary with $L^{r'}$ norm at most one, this implies that

$$\big\|h\big\|_{L^r} \leq \sup_{t>0} \bigg(\int_{\mathbf{R}^n} |F(x,t)|^p\,dx\bigg)^{\frac{1}{r}}. \tag{2.4.58}$$

Putting things together, we have

$$\begin{aligned}\big\||F|^*\big\|_{L^p} &\leq C_n' \big\|M(h)^{1/q}\big\|_{L^p} \\ &= C_n' \big\|M(h)\big\|_{L^r}^{1/q} \\ &\leq C_{n,p,q} \big\|h\big\|_{L^r}^{1/q}\end{aligned}$$