Given $\eta > 0$, we find a half-ball

$$B_{R_0} = \{(x,t) \in \mathbf{R}^{n+1}_+ : \ |x|^2 + t^2 < R_0^2\}$$

such that for $(x,t) \in \mathbf{R}^{n+1}_+ \setminus B_{R_0}$ we have

$$U(x,t) - V(x,t) \le \eta\,. \tag{2.4.54}$$

Suppose that this is possible. Since $U(x,0) = V(x,0)$, then (2.4.54) actually holds on the entire boundary of B_{R_0}. The function V is harmonic and U is subharmonic; thus $U - V$ is subharmonic. The maximum principle for subharmonic functions implies that (2.4.54) holds in the interior of B_{R_0}, and since it also holds on the exterior, it must be valid for all (x,t) with $x \in \mathbf{R}^n$ and $t \ge 0$. Since η was arbitrary, letting $\eta \to 0+$ implies (2.4.53).

We now prove that R_0 exists such that (2.4.54) is possible for $(x,t) \in \mathbf{R}^{n+1}_+ \setminus B_{R_0}$. Let $B((x,t),t/2)$ be the $(n+1)$-dimensional ball of radius $t/2$ centered at (x,t). The subharmonicity of $|F|^q$ is reflected in the inequality

$$|F(x,t)|^q \le \frac{1}{|B((x,t),\,t/2)|} \int_{B((x,t),t/2)} |F(y,s)|^q \, dy ds\,,$$

which by Hölder's inequality and the fact $p > q$ gives

$$|F(x,t)|^q \le \left(\frac{1}{|B((x,t),\,t/2)|} \int_{B((x,t),t/2)} |F(y,s)|^p \, dy ds \right)^{\frac{q}{p}}.$$

From this we deduce that

$$|F(x,t+\varepsilon)|^q \le \left[\frac{2^{n+1}/v_{n+1}}{(t+\varepsilon)^{n+1}} \int_{\frac{1}{2}(t+\varepsilon)}^{\frac{3}{2}(t+\varepsilon)} \int_{|y| \ge |x| - \frac{1}{2}(t+\varepsilon)} |F(y,s)|^p \, dy\, ds \right]^{\frac{q}{p}}. \tag{2.4.55}$$

If $t+\varepsilon \ge |x|$, using (2.4.42), we see that the expression on the right in (2.4.55) is bounded by $c' A^q (t+\varepsilon)^{-nq/p}$, and thus it can be made smaller than $\eta/2$ by taking $t \ge R_1 = \max\big(\varepsilon, (\eta/2c'A^q)^{-p/qn}\big)$. Since $R_1 \ge \varepsilon$, we must have $2t \ge t+\varepsilon \ge |x|$, which implies that $t \ge |x|/2$, and thus with $R_0' = \sqrt{5}R_1$, if $|(x,t)| > R_0'$ then $t \ge R_1$. Hence, the expression in (2.4.55) can be made smaller than $\eta/2$ for $|(x,t)| > R_0'$.

If $t+\varepsilon < |x|$ we estimate the expression on the right in (2.4.55) by

$$\left(\frac{2^{n+1}}{v_{n+1}} \frac{1}{(t+\varepsilon)^{n+1}} \int_{\frac{1}{2}(t+\varepsilon)}^{\frac{3}{2}(t+\varepsilon)} \left[\int_{|y| \ge \frac{1}{2}|x|} |F(y,s)|^p \, dy \right] ds \right)^{\frac{q}{p}},$$

and we notice that the preceding expression is bounded by

$$\left(\frac{3^{n+1}}{v_{n+1}} \int_{\frac{1}{2}\varepsilon}^{\infty} \left[\int_{|y| \ge \frac{1}{2}|x|} |F(y,s)|^p \, dy \right] \frac{ds}{s^{n+1}} \right)^{\frac{q}{p}}. \tag{2.4.56}$$