and such that

$$\frac{1}{c(s,n)} \le \frac{G_s(x)}{H_s(x)} \le c(s,n) \qquad \text{when } |x| \le 2, \tag{1.2.12}$$

where H_s is equal to

$$H_s(x) = \begin{cases} |x|^{s-n} + 1 + O(|x|^{s-n+2}) & \text{for } 0 < s < n, \\ \log \frac{2}{|x|} + 1 + O(|x|^2) & \text{for } s = n, \\ 1 + O(|x|^{s-n}) & \text{for } s > n, \end{cases}$$

and $O(t)$ is a function with the property $|O(t)| \le |t|$ for $t \ge 0$.

Now let z be a complex number with $\mathrm{Re}\, z > 0$. Then there exist finite positive constants $C'(\mathrm{Re}\, z, n)$ and $c'(\mathrm{Re}\, z, n)$ such that when $|x| \ge 2$, we have

$$|G_z(x)| \le \frac{C'(\mathrm{Re}\, z, n)}{|\Gamma(\frac{z}{2})|} e^{-\frac{|x|}{2}} \tag{1.2.13}$$

and when $|x| \le 2$, we have

$$|G_z(x)| \le \frac{c'(\mathrm{Re}\, z, n)}{|\Gamma(\frac{z}{2})|} \begin{cases} |x|^{\mathrm{Re}\, z - n} & \text{for } \mathrm{Re}\, z < n, \\ \log \frac{2}{|x|} + 1 & \text{for } \mathrm{Re}\, z = n, \\ 1 & \text{for } \mathrm{Re}\, z > n. \end{cases}$$

Proof. For $A > 0$ and z with $\mathrm{Re}\, z > 0$ we have the gamma function identity

$$A^{-\frac{z}{2}} = \frac{1}{\Gamma(\frac{z}{2})} \int_0^\infty e^{-tA} t^{\frac{z}{2}} \frac{dt}{t},$$

which we use to obtain

$$(1 + 4\pi^2 |\xi|^2)^{-\frac{z}{2}} = \frac{1}{\Gamma(\frac{z}{2})} \int_0^\infty e^{-t} e^{-\pi |2\sqrt{\pi t}\, \xi|^2} t^{\frac{z}{2}} \frac{dt}{t}.$$

Note that the preceding integral converges at both ends. Now take the inverse Fourier transform in ξ and use the fact that the function $e^{-\pi|\xi|^2}$ is equal to its Fourier transform (Example 2.2.9 in [156]) to obtain

$$G_z(x) = \frac{(2\sqrt{\pi})^{-n}}{\Gamma(\frac{z}{2})} \int_0^\infty e^{-t} e^{-\frac{|x|^2}{4t}} t^{\frac{z-n}{2}} \frac{dt}{t}.$$

This identity shows that G_z is smooth on $\mathbf{R}^n \setminus \{0\}$. Moreover, taking $z = s > 0$ proves that $G_s(x) > 0$ for all $x \in \mathbf{R}^n$. Consequently, $\|G_s\|_{L^1} = \int_{\mathbf{R}^n} G_s(x)\, dx = \widehat{G_s}(0) = 1$.