Proof. We begin the proof by observing that as a consequence of (2.4.23) we have

$$\|T(a)\|_{L^p} \le C_0 \tag{2.4.25}$$

for all a that are L^2-atoms for H^p. Indeed, (2.4.22) implies that for a given L^2 atom a for H^p, there is sequence $\varepsilon_k \downarrow 0$ such that

$$T(a) = \lim_{k\to\infty} T_{\varepsilon_k}(a) = \liminf_{k\to\infty} T_{\varepsilon_k}(a) \qquad \text{a.e.}$$

Then Fatou's lemma on L^p together with (2.4.23) imply (2.4.25).

Given $f \in H^p \cap L^2$, we write $f = \sum_{j=1}^\infty \lambda_j a_j$ in an atomic decomposition, where a_j are L^2-atoms for H^p, the series converges to f in H^p, and $\sum_{j=1}^\infty |\lambda_j|^p \le 2^p \|f\|_{H^p}^p$.

We observe that the sequence $\big\{ \sum_{j=1}^N \lambda_j T(a_j) \big\}_{N=1}^\infty$ is Cauchy in L^p since

$$\Big\| \sum_{j=N'}^N \lambda_j T(a_j) \Big\|_{L^p}^p \le \sum_{j=N'}^N |\lambda_j|^p C_0^p ,$$

which tends to zero as $N', N \to \infty$. Thus the sequence $\sum_{j=1}^N \lambda_j T(a_j)(x)$ converges in L^p to a well-defined L^p function. We set

$$\sum_{j=1}^\infty \lambda_j T(a_j) = L^p \text{ limit of } \sum_{j=1}^N \lambda_j T(a_j) .$$

To prove (2.4.24), we first prove an analogous result about T_ε, namely,

$$T_\varepsilon(f) = \sum_{j=1}^\infty \lambda_j T_\varepsilon(a_j) \qquad \text{a.e.} \tag{2.4.26}$$

where $\sum_{j=1}^\infty \lambda_j T_\varepsilon(a_j)$ denotes the L^p limit of the Cauchy sequence $\sum_{j=1}^N \lambda_j T_\varepsilon(a_j)$. We fix $\varepsilon, \delta > 0$. Then by the linearity of T_ε for each $L \in \mathbf{Z}^+$ we have

$$\begin{aligned}
&\Big| \Big\{ x \in \mathbf{R}^n : \ |T_\varepsilon(f)(x) - \sum_{j=1}^\infty \lambda_j T_\varepsilon(a_j)(x)| > \delta \Big\} \Big| \\
&\le \Big| \Big\{ x \in \mathbf{R}^n : \ |T_\varepsilon\big(\sum_{j=L+1}^\infty \lambda_j a_j \big)(x) - \sum_{j=L+1}^\infty \lambda_j T_\varepsilon(a_j)(x)| > \delta \Big\} \Big| \\
&\le \Big| \Big\{ x \in \mathbf{R}^n : |T_\varepsilon\big(\sum_{j=L+1}^\infty \lambda_j a_j \big)(x)| > \frac{\delta}{2} \Big\} \Big| + \Big| \Big\{ x \in \mathbf{R}^n : |\sum_{j=L+1}^\infty \lambda_j T_\varepsilon(a_j)(x)| > \frac{\delta}{2} \Big\} \Big| \\
&\le \frac{2^p}{\delta^p} \Big\| T_\varepsilon\big(\sum_{j=L+1}^\infty \lambda_j a_j \big) \Big\|_{L^p}^p + \frac{2^p}{\delta^p} \sum_{j=L+1}^\infty |\lambda_j|^p \big\| T_\varepsilon(a_j) \big\|_{L^p}^p .
\end{aligned} \tag{2.4.27}$$

By assumption (2.4.23) the second term in the sum in (2.4.27) is controlled by $C_0^p(\frac{2}{\delta})^p\sum_{j=L+1}^{\infty}|\lambda_j|^p$ which tends to zero as $L\to\infty$.

To show the same conclusion for the first sum in (2.4.27) we recall the grand maximal function

$$\mathscr{M}_N(f)(x)=\sup_{\varphi\in\mathscr{F}_N}\sup_{t>0}\sup_{\substack{y\in\mathbf{R}^n\\|y-x|<t}}|(\varphi_t*f)(y)|$$

where

$$\mathscr{F}_N=\Big\{\varphi\in\mathscr{S}(\mathbf{R}^n):\ \int_{\mathbf{R}^n}(1+|x|)^N\sum_{|\alpha|\le N+1}|\partial^\alpha\varphi(x)|\,dx\le 1\Big\}.$$

The function ζ_ε lies in $\mathscr{S}(\mathbf{R}^n)$; thus there is a constant $c_{\varepsilon,N}$ such that $c_{\varepsilon,N}\zeta_\varepsilon$ lies in $\mathscr{F}_N$. Then we have

$$|T_\varepsilon\big(\sum_{j=L+1}^{\infty}\lambda_j a_j\big)|\le\frac{1}{c_{\varepsilon,N}}\mathscr{M}_N\big(f-\sum_{j=1}^{L}\lambda_j a_j\big).$$

Taking L^p quasi-norms we obtain

$$\Big\|T_\varepsilon\big(\sum_{j=L+1}^{\infty}\lambda_j a_j\big)\Big\|_{L^p}^p\le\frac{1}{c_{\varepsilon,N}^p}\Big\|\mathscr{M}_N\big(f-\sum_{j=1}^{L}\lambda_j a_j\big)\Big\|_{L^p}^p\le\frac{C_{n,p}^p}{c_{\varepsilon,N}^p}\Big\|f-\sum_{j=1}^{L}\lambda_j a_j\Big\|_{H^p}^p,$$

and since $\sum_{j=1}^{L}\lambda_j a_j\to f$ in H^p as $L\to\infty$, we deduce that the first sum in (2.4.27) tends to zero as $L\to\infty$. This proves that for any $\varepsilon,\delta>0$ we have

$$\Big|\Big\{x\in\mathbf{R}^n:\ |T_\varepsilon(f)(x)-\sum_{j=1}^{\infty}\lambda_j T_\varepsilon(a_j)(x)|>\delta\Big\}\Big|=0;$$

hence (2.4.26) holds.

Next, we claim that $\sum_{j=1}^{\infty}\lambda_j T_\varepsilon(a_j)\to\sum_{j=1}^{\infty}\lambda_j T(a_j)$ in measure as $\varepsilon\to 0$. Indeed, given $\delta>0$, write

$$\begin{aligned}
&\Big|\Big\{\sum_{j=1}^{\infty}\lambda_j T_\varepsilon(a_j)-\sum_{j=1}^{\infty}\lambda_j T(a_j)|>\delta\Big\}\Big|\\
&\le\Big|\Big\{\Big|\sum_{j=1}^{L}\lambda_j\big(T_\varepsilon(a_j)-T(a_j)\big)\Big|>\frac{\delta}{2}\Big\}\Big|+\Big|\Big\{\Big|\sum_{j=L+1}^{\infty}\lambda_j T_\varepsilon(a_j)-\sum_{j=L+1}^{\infty}\lambda_j T(a_j)\Big|>\frac{\delta}{2}\Big\}\Big|\\
&\le\frac{2^2}{\delta^2}\Big\|\sum_{j=1}^{L}\lambda_j\big(T_\varepsilon(a_j)-T(a_j)\big)\Big\|_{L^2}^2+\frac{2^p}{\delta^p}\sum_{j=L+1}^{\infty}|\lambda_j|^p\Big[\big\|T_\varepsilon(a_j)\big\|_{L^p}^p+\big\|T(a_j)\big\|_{L^p}^p\Big]\\
&\le\frac{2^2}{\delta^2}\Big\|T_\varepsilon\big(\sum_{j=1}^{L}\lambda_j a_j\big)-T\big(\sum_{j=1}^{L}\lambda_j a_j\big)\Big\|_{L^2}^2+\frac{2^{p+1}C_0^p}{\delta^p}\sum_{j=L+1}^{\infty}|\lambda_j|^p,
\end{aligned}\tag{2.4.28}$$