Proof. Let Ψ be a Schwartz function whose Fourier transform is supported in the annulus $1-\frac{1}{7} \le |\xi| \le 2$ and that satisfies

$$\sum_{j\in\mathbf{Z}} \widehat{\Psi}(2^{-j}\xi) = 1\,, \qquad \xi \neq 0\,.$$

Pick a Schwartz function ζ whose Fourier transform $\widehat{\zeta}$ is supported in the annulus $\frac{1}{4} < |\xi| < 8$ and that is equal to one on the support of $\widehat{\Psi}$. Let W be the tempered distribution that coincides with K on $\mathbf{R}^n \setminus \{0\}$ so that $T(f) = f * W$. Then we have $\zeta_{2^{-j}} * \Psi_{2^{-j}} = \Psi_{2^{-j}}$ for all j and hence

$$\begin{aligned} \big\|\Delta_j(T(f))\big\|_{L^p} &= \big\|\zeta_{2^{-j}} * \Psi_{2^{-j}} * W * f\big\|_{L^p} \\ &\le \big\|\zeta_{2^{-j}} * W\big\|_{L^1} \big\|\Delta_j(f)\big\|_{L^p}, \end{aligned} \tag{2.4.11}$$

since $1 \le p \le \infty$. It is not hard to check that the function $\zeta_{2^{-j}}$ is in H^1 with norm independent of j. Therefore, $\zeta_{2^{-j}}$ is in H^1. Using Theorem 2.4.1, we conclude that

$$\big\|T(\zeta_{2^{-j}})\big\|_{L^1} = \big\|\zeta_{2^{-j}} * W\big\|_{L^1} \le C \big\|\zeta_{2^{-j}}\big\|_{H^1} = C'\,.$$

Inserting this in (2.4.11), multiplying by $2^{j\alpha}$, and taking ℓ^q quasi-norms, we obtain the required conclusion. □

2.4.3 Singular Integrals on $H^p(\mathbf{R}^n)$

It is possible to extend Theorem 2.4.1 to $H^p(\mathbf{R}^n)$ for $p < 1$, provided the kernel K has additional smoothness.

For the purposes of this subsection, we fix a $\mathscr{C}^\infty$ function $K(x)$ on $\mathbf{R}^n \setminus \{0\}$. We suppose that there is a positive integer N (to be specified later) such that

$$|\partial^\beta K(x)| \le A\,|x|^{-n-|\beta|} \qquad \text{for all } |\beta| \le N \tag{2.4.12}$$

and that

$$\sup_{0<R_1<R_2<\infty} \left| \int_{R_1<|x|<R_2} K(x)\,dx \right| \le A\,, \tag{2.4.13}$$

for some $A < \infty$.

We fix a nonnegative smooth function η on $\mathbf{R}^n$ that vanishes in the unit ball of $\mathbf{R}^n$ and is equal to 1 outside the ball $B(0,2)$ and for $0 < \varepsilon < 1/2$ we define ~~the smoothly truncated family of kernels~~

$$\cancel{K^{(\varepsilon)}(x) = K(x)\eta(x/\varepsilon)}$$

~~and~~ the doubly smoothly truncated family of kernels

$$K_{(\varepsilon)}(x) = K(x)\eta(x/\varepsilon) - K(x)\eta(\varepsilon x)\,.$$

Condition (2.4.12) with $\beta = 0$ and (2.4.13) imply that

$$\left| \int_{|x|\le 1} K(x)\eta(x/\varepsilon)\,dx \right| \le (1 + \omega_{n-1}\log 2)A$$

for all $\varepsilon < 1/2$; hence there exists a sequence $\varepsilon_j < 1/2$ with $\varepsilon_j \downarrow 0$ as $j \to \infty$ such that the following limit exists:

$$\lim_{j\to\infty} \int_{|x|\le 1} K(x)\eta(x/\varepsilon_j)\,dx = L_0.$$

We now define W in $\mathscr{S}'(\mathbf{R}^n)$ by setting

$$\begin{aligned} \langle W, \varphi\rangle &= \lim_{j\to\infty} \int_{\mathbf{R}^n} K_{(\varepsilon_j)}(x)\varphi(x)\,dx \qquad (2.4.14)\\ &= L_0\varphi(0) + \int_{|x|\le 1} K(x)(\varphi(x) - \varphi(0))\,dx + \int_{|x|\ge 1} K(x)\varphi(x)\,dx \end{aligned}$$

for φ in $\mathscr{S}$. It is quite easy to verify that the preceding expression is bounded by a constant multiple of a finite sum of Schwartz seminorms of φ. Note that this distribution[4] depends on the number L_0 and hence on the bump η.

We define the associated doubly smoothly truncated singular integral by setting

$$T_{(\varepsilon)}(\varphi)(x) = \int_{\mathbf{R}^n} K_{(\varepsilon)}(y)\varphi(x-y)\,dy \qquad (2.4.15)$$

for Schwartz functions φ on $\mathbf{R}^n$.

We also define an operator T given by convolution with W by setting

$$T(\varphi) = \lim_{j\to\infty} T_{(\varepsilon_j)}(\varphi) = \varphi * W \qquad (2.4.16)$$

for $\varphi \in \mathscr{S}(\mathbf{R}^n)$. Observe that W coincides with K on $\mathbf{R}^n \setminus \{0\}$, since if φ is supported in $|x| \ge t_0 > 0$, (2.4.14) implies that the action of W on $\varphi \in \mathscr{S}$ coincides with that of $K^{(\varepsilon_j)}$ on φ when $\varepsilon_j < t_0/2$. Condition (2.4.12) with $|\beta| = 1$ implies

$$\sup_{y\neq 0} \int_{|x|\ge 2|y|} |K(x-y) - K(x)|\,dx \le cA\,; \qquad (2.4.17)$$

hence Theorem 5.4.1 in [156] yields the L^2 boundedness of T. Note that (2.4.17) also holds for $K_{(\varepsilon)}$ in place of K uniformly in ε; thus again by Theorem 5.4.1 in [156] the operators $T_{(\varepsilon)}$ are uniformly bounded on $L^2(\mathbf{R}^n)$.

[4] Alternatively, we could have defined W as an element of $\mathscr{S}'(\mathbf{R}^n)/\mathscr{P}(\mathbf{R}^n)$ acting on functions $\varphi \in \mathscr{S}_0$; in this case W would have been independent of L_0 and η.

We summarize these and other observations about $K_{(\varepsilon)}$, $T_{(\varepsilon)}$, and T.

(i) The kernels $K_{(\varepsilon)}$ satisfy the same estimates as K uniformly in ε with constant A' in place of A, where A' is comparable to A.
(ii) $T_{(\varepsilon)}$ are uniformly bounded on L^2.
(iii) $T_{(\varepsilon_j)}(g)$ tends to $T(g)$ in L^2 for any $g \in L^2(\mathbf{R}^n)$.
(iv) T is L^2 bounded with norm $\|\widehat{W}\|_{L^\infty} \le CA$.
(v) For any $f \in H^p$, $T(f)$ is a well-defined element of $\mathscr{S}'$.

We have already explained assertions (i) and (ii) and (iv).
We explain (iii). Theorem 5.3.4 in [156] gives that for all $g \in L^2$ we have

$$\sup_{\varepsilon>0} |T_{(\varepsilon)}(g)| \le M(T(g)) + C_n A\, M(g)\,;$$

hence the maximal operator $T^{(**)}(g) = \sup_{\varepsilon>0} |T_{(\varepsilon)}(g)|$ is L^2 bounded. Moreover, as an easy consequence of (2.4.14), for each $\varphi \in \mathscr{S}$ we have $T_{(\varepsilon_j)}(\varphi) \to T(\varphi)$ pointwise everywhere. In view of Theorem 2.1.14 in [156], for every $g \in L^2(\mathbf{R}^n)$ we have $T_{(\varepsilon_j)}(g) - T(g) \to 0$ a.e. as $j \to \infty$. Since

$$|T_{(\varepsilon_j)}(g) - T(g)| \le 2\,T^{(**)}(g) \in L^2\,,$$

the Lebesgue dominated convergence theorem yields that $T_{(\varepsilon_j)}(g) - T(g) \to 0$ in L^2.

To verify the validity of (v) we write $W = W_0 + K_\infty$, where $W_0 = \Phi W$ and $K_\infty = (1-\Phi)K$, where Φ is a smooth function equal to one on the ball $B(0,1)$ and vanishing outside the ball $B(0,2)$. Then for f in $H^p(\mathbf{R}^n)$, $0 < p \le 1$, we define a tempered distribution $T(f) = W * f$ by setting

$$\big\langle T(f), \phi \big\rangle = \big\langle f, \phi * \widetilde{W_0} \big\rangle + \big\langle \widetilde{\phi} * f, \widetilde{K_\infty} \big\rangle \tag{2.4.18}$$

for ϕ in $\mathscr{S}(\mathbf{R}^n)$. (Here $\widetilde{\varphi}(x) = \varphi(-x)$ for functions and analogously for distributions.) The function $\phi * \widetilde{W_0}$ is in $\mathscr{S}$, so the action of f on it is well defined. Also $\widetilde{\phi} * f$ is in L^1 (see Proposition 2.1.9), while $\widetilde{K_\infty}$ is in L^∞; hence the second term on the right in (2.4.18) represents an absolutely convergent integral. Moreover, in view of Theorem 2.3.20 in [156] and Corollary 2.1.9, both terms on the right in (2.4.18) are controlled by a finite sum of seminorms $\rho_{\alpha,\beta}(\phi)$ (cf. Definition 2.2.1 in [156]). This defines $T(f)$ as a tempered distribution for every $f \in H^p$.

The following is an extension of Theorem 2.4.1 for $p < 1$.

Theorem 2.4.3. *Let* $0 < p < 1$ *and* $N = [\frac{n}{p} - n] + 1$. *Let* K *be a* $\mathscr{C}^\infty$ *function on* $\mathbf{R}^n \setminus \{0\}$ *that satisfies* (2.4.13) *and* (2.4.12) *for some* $A < \infty$ *for all multi-indices* $|\beta| \le N$ *and all* $x \ne 0$. *Let* W *be a tempered distribution that coincides with* K *on* $\mathbf{R}^n \setminus \{0\}$, *as defined in* (2.4.14). *Then there is a constant* $C_{n,p}$ *such that for all* $f \in H^p$ *the distribution* $T(f)$ *defined in* (2.4.18) *coincides with an* L^p *function that satisfies*

$$\big\|T(f)\big\|_{L^p} \le C_{n,p} A \big\|f\big\|_{H^p}\,.$$

Proof. The proof of this theorem is based on the atomic decomposition of H^p.

We first take $f = a$ to be an L^2-atom for H^p, and without loss of generality we may assume that a is supported in a cube Q centered at the origin. We let Q^* be the cube with side length $2\sqrt{n}\,\ell(Q)$, where $\ell(Q)$ is the side length of Q. We have

$$\begin{aligned}
\left(\int_{Q^*} |T(a)(x)|^p\,dx\right)^{\frac{1}{p}} &\le C|Q^*|^{\frac{1}{p}-\frac{1}{2}}\left(\int_{Q^*} |T(a)(x)|^2\,dx\right)^{\frac{1}{2}} \\
&\le C''A|Q|^{\frac{1}{p}-\frac{1}{2}}\left(\int_{Q} |a(x)|^2\,dx\right)^{\frac{1}{2}} \\
&\le C_n A\,|Q|^{\frac{1}{p}-\frac{1}{2}}|Q|^{\frac{1}{2}-\frac{1}{p}} \\
&= C_n A\,,
\end{aligned}$$

where we used that T is L^2 bounded with norm at most a constant multiple of A.

For $x \notin Q^*$ and $y \in Q$, we have $|x| \ge 2|y|$, and thus $x-y$ stays away from zero and $K(x-y)$ is well defined. We have

$$T(a)(x) = \int_Q K(x-y)\,a(y)\,dy\,.$$

Recall that $N = [\frac{n}{p} - n] + 1$. Using the cancellation of atoms in H^p, we deduce

$$\begin{aligned}
T(a)(x) &= \int_Q a(y)K(x-y)\,dy \\
&= \int_Q a(y)\left[K(x-y) - \sum_{|\beta|\le N-1} (\partial^\beta K)(x)\frac{(-y)^\beta}{\beta!}\right]dy \\
&= \int_Q a(y)\left[\int_0^1 (N+1)(1-\theta)^N \sum_{|\beta|=N} (\partial^\beta K)(x-\theta y)\frac{(-y)^\beta}{\beta!}\,d\theta\right]dy\,.
\end{aligned}$$

~~for some $0 \le \theta_y \le 1$~~ by Appendix I in [156]. The fact that $|x| \ge 2|y|$ implies that $|x-\theta y| \ge \frac{1}{2}|x|$ and using (2.4.12) we obtain the estimate

$$|T(a)(x)| \le c_{n,N}\frac{A}{|x|^{N+n}}\int_Q |a(y)|\,|y|^N\,dy\,,$$

from which it follows that for $x \notin Q^*$ we have

$$|T(a)(x)| \le c_{n,p}\frac{A}{|x|^{N+n}}|Q|^{1+\frac{N}{n}-\frac{1}{p}}$$

via a calculation using Hölder's inequality and the fact that $\|a\|_{L^q} \le |Q|^{\frac{1}{q}-\frac{1}{p}}$. Integrating over $(Q^*)^c$, we obtain that

$$\left(\int_{(Q^*)^c} |T(a)(x)|^p dx\right)^{\frac{1}{p}} \le c_{n,p}A\,|Q|^{1+\frac{N}{n}-\frac{1}{p}}\left(\int_{(Q^*)^c}\frac{1}{|x|^{p(N+n)}}\,dx\right)^{\frac{1}{p}} \le c'_{n,p}A\,.$$