$$
\begin{aligned}
\big|\{C_{n,s,1} M(f)^{\frac{n-s}{n}} \|f\|_{L^1}^{\frac{s}{n}} > \lambda\}\big| &= \Big|\Big\{M(f) > \Big(\frac{\lambda}{C_{n,s,1}\|f\|_{L^1}^{\frac{s}{n}}}\Big)^{\frac{n}{n-s}}\Big\}\Big| \\
&\le 3^n \Big(\frac{C_{n,s,1}\|f\|_{L^1}^{\frac{s}{n}}}{\lambda}\Big)^{\frac{n}{n-s}} \|f\|_{L^1} \\
&= C(n,s)\Big(\frac{\|f\|_{L^1}}{\lambda}\Big)^{\frac{n}{n-s}}.
\end{aligned}
$$

This estimate says that $\mathcal{I}_s$ maps $L^1(\mathbf{R}^n)$ to weak $L^{\frac{n}{n-s}}(\mathbf{R}^n)$. □

1.2.2 Bessel Potentials

While the behavior of the kernels $|x|^{-n+s}$ as $|x| \to 0$ is well suited to their smoothing properties, their decay as $|x| \to \infty$ gets worse as s increases. We can slightly adjust the Riesz potentials so that we maintain their essential behavior near zero but achieve exponential decay at infinity. The simplest way to achieve this is by replacing the *nonnegative* operator $-\Delta$ by the *strictly positive* operator $I - \Delta$. Here the terms *nonnegative* and *strictly positive*, as one may have surmised, refer to the Fourier multipliers of these operators.

Definition 1.2.4. Let z be a complex number satisfying $0 < \operatorname{Re} z < \infty$. The *Bessel potential operator* of order z is

$$\mathcal{J}_z = (I - \Delta)^{-z/2}.$$

This operator acts on functions f as follows:

$$\mathcal{J}_z(f) = \big(\widehat{f}\,\widehat{G_z}\big)^\vee = f * G_z\,,$$

where

$$G_z(x) = \big((1 + 4\pi^2|\xi|^2)^{-z/2}\big)^\vee(x)\,.$$

The Bessel potential is obtained by replacing $4\pi^2|\xi|^2$ in the Riesz potential by the smooth term $1 + 4\pi^2|\xi|^2$. This adjustment creates smoothness, which yields rapid decay for G_z at infinity. The next result quantifies the behavior of G_z near zero and near infinity.

Proposition 1.2.5. *Let z be a complex number with* $\operatorname{Re} z > 0$. *Then the function G_z is smooth on $\mathbf{R}^n \setminus \{0\}$. Moreover, if s is real, then G_s is strictly positive, $\|G_s\|_{L^1} = 1$, and there exist positive finite constants $C(s,n)$, $c(s,n)$ such that*

$$G_s(x) \le C(s,n) e^{-\frac{|x|}{2}} \qquad \text{when } |x| \ge 2 \tag{1.2.11}$$