Since $\|s\|_{\dot{f}_p^{0,2}} \le C_{n,p}\big\|f\big\|_{H^p}$, Theorem 2.3.4 implies that

$$\Big(\sum_{j=1}^{\infty} |\lambda_j|^p\Big)^{\frac{1}{p}} \le c'\big\|f\big\|_{H^p}. \tag{2.3.29}$$

For $j = 1,2,\dots$ set

$$A_j = \sum_{\mu\in\mathbf{Z}}\bigg(\sum_{Q\in\mathscr{D}_\mu} r_{j,Q}a_Q\bigg)$$

where the series converges in H^p (see Theorem 2.3.4) and the H^p quasi-norm of A_j is bounded by a constant in view of (2.3.4), since $\|r_j\|_{\dot{f}_p^{0,2}} \le 1$. Using again (2.3.4) in Theorem 2.3.4 we obtain

$$\begin{aligned}
\Big\|\sum_{j=1}^{N}\lambda_j A_j - f\Big\|_{H^p}^p &\le C_{n,p}^p\Big\|\sum_{j=1}^{N}\lambda_j r_j - s\Big\|_{\dot{f}_p^{0,2}}^p \\
&\le C_{n,p}^p\Big\|\sum_{j=N+1}^{\infty}|\lambda_j|\,|r_j|\Big\|_{\dot{f}_p^{0,2}}^p \\
&\le C_{n,p}^p\sum_{j=N+1}^{\infty}|\lambda_j|^p \to 0\,,
\end{aligned}$$

as $N\to\infty$, where the last inequality follows from Exercise 2.3.5(a).

Next we show that each A_j is a fixed multiple of an L^2-atom for H^p. Let us fix an index j. By the definition of the ∞-atom for $\dot{f}_p^{0,2}$, there exists a dyadic cube Q_0^j such that $r_{j,Q} = 0$ for all dyadic cubes Q not contained in Q_0^j. Then the support of each a_Q is contained in $3Q$, hence in $3Q_0^j$. This implies that the function A_j is supported in $3Q_0^j$. The same is true for the function $g^{0,2}(r_j)$ defined in (2.3.2). Using this fact, we have

$$\begin{aligned}
\big\|A_j\big\|_{L^2} &\approx \big\|A_j\big\|_{\dot{F}_2^{0,2}} \\
&\le c\,\big\|r_j\big\|_{\dot{f}_2^{0,2}} \\
&= c\,\big\|g^{0,2}(r_j)\big\|_{L^2} \\
&\le c\,\big\|g^{0,2}(r_j)\big\|_{L^\infty}|3Q_0^j|^{\frac{1}{2}} \\
&\le c\,|3Q_0^j|^{-\frac{1}{p}+\frac{1}{2}}\,.
\end{aligned}$$

Since

$$g^{0,2}(r_j) = \Big(\sum_{Q\in\mathscr{D}}|Q|^{-1}|r_{j,Q}|^2\chi_Q\Big)^{\frac{1}{2}}$$

the estimate $\|g^{0,2}(r_j)\|_{L^2} \le |3Q_0^j|^{-\frac{1}{p}+\frac{1}{2}}$ we proved implies that

$$\sum_{Q\in\mathscr{D}} |r_{j,Q}|^2 < \infty.$$

Let $M' < M$ be positive integers. Then

$$\begin{aligned}\Big\|\sum_{M'<|\mu|\le M}\sum_{Q\in\mathscr{D}_\mu} r_{j,Q}a_Q\Big\|_{L^1} &\le |3Q_0^j|^{\frac{1}{2}}\Big\|\sum_{M'<|\mu|\le M}\sum_{Q\in\mathscr{D}_\mu} r_{j,Q}a_Q\Big\|_{L^2}\\ &= |3Q_0^j|^{\frac{1}{2}}\Big(\sum_{M'<|\mu|\le M}\sum_{Q\in\mathscr{D}_\mu} |r_{j,Q}|^2\Big)^{\frac{1}{2}} \to 0\end{aligned}$$

as $M',M\to\infty$. Therefore the sequence $\sum_{|\mu|\le M}\sum_{Q\in\mathscr{D}_\mu} r_{j,Q}a_Q$ is Cauchy in L^1 and hence it converges in L^1. But this sequence converges in H^p to A_j by Theorem 2.3.4, so finally it converges to A_j in L^1.

The fact that $A_j = \sum_{\mu\in\mathbf{Z}}\sum_{Q\in\mathscr{D}_\mu} r_{j,Q}a_Q$ with convergence in L^1 allows us to deduce that vanishing moments of a_Q pass on to A_j. We conclude that each A_j is a fixed multiple of an L^2-atom for H^p. The $\ge$ direction in (2.3.28) now follows from (2.3.29), given that we have now established all the remaining properties. □

Remark 2.3.13. Property (c) in Definition 2.3.10 can be replaced by

$$\int x^\gamma A(x)\,dx = 0 \qquad \text{for all multi-indices } \gamma \text{ with } |\gamma|\le L,$$

for any $L \ge [\frac{n}{p}-n]$, and the atomic decomposition of H^p holds unchanged. In fact, in the proof of Theorem 2.3.12 we may take $L \ge [\frac{n}{p}-n]$ instead of $L = [\frac{n}{p}-n]$ and then apply Theorem 2.3.4 for this L. Note that Theorem 2.3.4 was valid for all $L \ge [\frac{n}{p}-n]$. This observation turns out to be quite useful in certain applications.

Exercises

2.3.1. (a) Given $N\in\mathbf{Z}^+$, prove that there exists a smooth function Θ supported in the unit ball $|x|\le 1$ such that $\int_{\mathbf{R}^n} x^\gamma \Theta(x)\,dx = 0$ for all multi-indices γ with $|\gamma|\le N$ and such that $|\widehat{\Theta}(\xi)|\ge\frac{1}{2}$ for all ξ in the annulus $\frac{1}{2}\le|\xi|\le 2$.
(b) Prove there exists a Schwartz function Ψ whose Fourier transform is supported in the annulus $\frac{1}{2}\le|\xi|\le 2$, with $|\widehat{\Psi}(\xi)|\ge c>0$ in the smaller annulus $\frac{3}{5}\le|\xi|\le\frac{5}{3}$, and which satisfies for all $\xi\in\mathbf{R}^n\setminus\{0\}$

$$\sum_{j\in\mathbf{Z}} \widehat{\Psi}(2^{-j}\xi)\widehat{\Theta}(2^{-j}\xi) = 1\,.$$

$\big[$*Hint:* Part (a): Let θ be an even real-valued smooth function supported in the ball $|x|\le 1$ and such that $\widehat{\theta}(0)=1$. Then for some $\varepsilon\in(0,\frac{1}{2})$ we have $\widehat{\theta}(\xi)\ge\frac{1}{2}$ for all