where $\{\lambda_1,\lambda_2,\dots\}=\{\lambda_{k,J}:\ (k,J)\in\mathscr{U}\}$ and $\{r_1,r_2,\dots\}=\{r(k,J):\ (k,J)\in\mathscr{U}\}$. As observed the sum in (2.3.21) has the property that for each $Q\in\mathscr{D}$, there is at most one $k\in\mathbf{Z}$ and at most one $J\in\mathscr{B}_k$ such that $\lambda_{k,J}\,r(k,J)_Q=t(k,J)_Q$ is nonzero. Thus for each $Q\in\mathscr{D}$, at most one term in the sum $\sum_{j=1}^{\infty}\lambda_j r_{j,Q}$ is nonzero; in particular, this series is absolutely convergent.

Finally, we estimate the sum of the pth power of the coefficients $\lambda_{k,J}$. We have

$$\begin{aligned}
\sum_{j=1}^{\infty}|\lambda_j|^p &= \sum_{k\in\mathbf{Z}}\sum_{J\in\mathscr{B}_k}\lambda_{k,J}^p\\
&= \sum_{k\in\mathbf{Z}}2^{(k+1)p}\sum_{J\in\mathscr{B}_k}|J|\\
&\le 2^p\sum_{k\in\mathbf{Z}}2^{kp}\Big|\bigcup_{Q\in\mathscr{A}_k}Q\Big|\\
&= 2^p\sum_{k\in\mathbf{Z}}2^{k(p-1)}2^k|\{x\in\mathbf{R}^n:\ g^{\alpha,q}(s)(x)>2^k\}|\\
&\le 2^p\sum_{k\in\mathbf{Z}}\int_{2^k}^{2^{k+1}}2^{k(p-1)}|\{x\in\mathbf{R}^n:\ g^{\alpha,q}(s)(x)>\tfrac{\lambda}{2}\}|\,d\lambda\\
&\le 2^p\sum_{k\in\mathbf{Z}}\int_{2^k}^{2^{k+1}}\lambda^{p-1}|\{x\in\mathbf{R}^n:\ g^{\alpha,q}(s)(x)>\tfrac{\lambda}{2}\}|\,d\lambda\\
&= \frac{2^{2p}}{p}\big\|g^{\alpha,q}(s)\big\|_{L^p}^p\\
&= \frac{2^{2p}}{p}\|s\|_{\dot{f}_p^{\alpha,q}}^p .
\end{aligned}$$

Taking the pth root yields (2.3.15). The proof of the theorem is now complete. □

We now deduce a corollary concerning a new characterization of the space $\dot{f}_p^{\alpha,q}$.

Corollary 2.3.7. *Suppose $\alpha\in\mathbf{R}$, $0<p\le 1$, and $p\le q<\infty$. Then for a given sequence $s\in\dot{f}_p^{\alpha,q}$ we have the following equivalence:*

$$\|s\|_{\dot{f}_p^{\alpha,q}}\approx\inf\left\{\Big(\sum_{j=1}^{\infty}|\lambda_j|^p\Big)^{\frac1p}:\ \lim_{N\to\infty}\Big\|s-\sum_{j=1}^{N}\lambda_j r_j\Big\|_{\dot{f}_p^{\alpha,q}}=0,\ r_j\ \textit{are}\ \infty\textit{-atoms for}\ \dot{f}_p^{\alpha,q}\right\}.$$

Remark 2.3.8. Notice that $\dot{f}_p^{\alpha,q}$ is complete (Exercise 2.3.5(b)), so if r_j are ∞-atoms for $\dot{f}_p^{\alpha,q}$, if $(\sum_{j=1}^{\infty}|\lambda_j|^p)^{\frac1p}<\infty$ and if

$$\Big\|s-\sum_{j=1}^{N}\lambda_j r_j\Big\|_{\dot{f}_p^{\alpha,q}}\to 0$$

as $N\to\infty$, then s must be an element of $\dot{f}_p^{\alpha,q}$.