In view of properties (a) and (c) of Definition 2.3.3, for every $M > 0$ there is a constant $C(n,M,L)$ such that every smooth L-atom a_Q supported in $3Q$ with center c_Q and side length $\ell(Q)$ satisfies

$$|\partial^\gamma a_Q(x)| \le C(n,M,L)\ell(Q)^{\frac{n}{2}} \frac{\ell(Q)^{-n-|\gamma|}}{\big(1+\frac{|x-c_Q|}{\ell(Q)}\big)^M} \tag{2.3.3}$$

for all $x \in \mathbf{R}^n$ and for all multi-indices γ with $|\gamma| \le L+1$.

We now prove a theorem stating that elements of $\dot{F}_p^{\alpha,q}$ can be decomposed as sums of smooth atoms.

Theorem 2.3.4. *Let $0 < p,q < \infty$, $\alpha \in \mathbf{R}$, and let*

$$L = \big[\max\big(n\max(1,\tfrac{1}{p},\tfrac{1}{q}) - n - \alpha, \alpha\big)\big].$$

Then there is a constant $C_{n,p,q,\alpha}$ such that for every sequence of smooth L-atoms $\{a_Q\}_{Q\in\mathscr{D}}$ and every sequence of complex scalars $\{s_Q\}_{Q\in\mathscr{D}}$ in $\dot{f}_p^{\alpha,q}$ we have that the series $\sum_{\mu\in\mathbf{Z}}\big(\sum_{Q\in\mathscr{D}_\mu} s_Q a_Q\big)$ converges in $\dot{F}_p^{\alpha,q}(\mathbf{R}^n)$ to an element f of $\dot{F}_p^{\alpha,q}(\mathbf{R}^n)$ with quasi-norm

$$\big\|f\big\|_{\dot{F}_p^{\alpha,q}} \le C_{n,p,q,\alpha}\big\|\{s_Q\}_Q\big\|_{\dot{f}_p^{\alpha,q}}. \tag{2.3.4}$$

Conversely, for any $L \in \mathbf{Z}^+$, there is a constant $C'_{n,p,q,\alpha,L}$ such that given any distribution f in $\dot{F}_p^{\alpha,q}$, there exist a sequence of smooth L-atoms $\{a_Q\}_{Q\in\mathscr{D}}$ and a sequence of complex scalars $\{s_Q\}_{Q\in\mathscr{D}}$ such that the series $\sum_{\mu\in\mathbf{Z}}\big(\sum_{Q\in\mathscr{D}_\mu} s_Q a_Q\big)$ converges to f in $\dot{F}_p^{\alpha,q}(\mathbf{R}^n)$ and

$$\big\|\{s_Q\}_Q\big\|_{\dot{f}_p^{\alpha,q}} \le C'_{n,p,q,\alpha,L}\big\|f\big\|_{\dot{F}_p^{\alpha,q}}. \tag{2.3.5}$$

We observe that for any given x the expression $\sum_{Q\in\mathscr{D}_\mu} s_Q a_Q(x)$ is a finite sum with at most 3^n summands, so the convergence concerns the series in μ.

Proof. We prove the first assertion of the theorem. We let Δ_j^Ψ be the Littlewood–Paley operator associated with a Schwartz function Ψ whose Fourier transform is compactly supported away from the origin in $\mathbf{R}^n$. Let a_Q be a smooth L-atom supported in a cube $3Q$ with center c_Q and let the side length of Q be $\ell(Q) = 2^{-\mu}$. It follows from (2.3.3) that a_Q satisfies

$$|\partial_y^\gamma a_Q(y)| \le C_{N',n} 2^{-\frac{\mu n}{2}} \frac{2^{\mu|\gamma|+\mu n}}{(1+2^\mu|y-c_Q|)^{N'}} \tag{2.3.6}$$

for all $N' > 0$ and for all multi-indices γ satisfying $|\gamma| \le L+1$. Moreover, the function $y \mapsto \Psi_{2^{-j}}(y-x)$ satisfies

$$|\partial_y^\beta \Psi_{2^{-j}}(y-x)| \le C_{N',n,\beta} \frac{2^{j|\beta|+jn}}{(1+2^j|y-x|)^{N'}} \tag{2.3.7}$$

for all $N' > 0$ and for all multi-indices β.