or, equivalently,

$$\big\|\mathcal{I}_s(f)\big\|_{L^q(\mathbf{R}^n)} \le C(p,q,n,s)\lambda^{\frac{n}{q}-\frac{n}{p}+\mathrm{Re}\,s}\big\|f\big\|_{L^p(\mathbf{R}^n)}. \tag{1.2.6}$$

If $\frac{1}{p} > \frac{1}{q} + \frac{\mathrm{Re}\,s}{n}$, then we let $\lambda \to \infty$ in (1.2.6), whereas if $\frac{1}{p} < \frac{1}{q} + \frac{\mathrm{Re}\,s}{n}$, then we let $\lambda \to 0$ in (1.2.6). In both cases we obtain that $\mathcal{I}_s(f) = 0$ for all Schwartz functions f, but this is obviously not the case for the function $f(x) = e^{-\pi|x|^2}$. It follows that (1.2.4) must necessarily hold.

This example provides an excellent paradigm of situations where the homogeneity (or the dilation structure) of an operator dictates a relationship on the indices p and q for which it (may) map L^p to L^q.

As we saw in Remark 1.2.2, if the Riesz potentials map L^p to L^q for some p,q, then we must have $q > p$. Such operators that improve the integrability of a function are called *smoothing*. The importance of the Riesz potentials lies in the fact that they are indeed smoothing operators. This is the essence of the *Hardy–Littlewood–Sobolev theorem on fractional integration*, which we now formulate and prove. Since

$$|\mathcal{I}_s(f)| \le \mathcal{I}_{\mathrm{Re}\,s}(|f|),$$

one may restrict the study of $\mathcal{I}_s(f)$ to nonnegative functions f and $s > 0$.

Theorem 1.2.3. *Let s be a real number, with $0 < s < n$, and let $1 < p < q < \infty$ satisfy*

$$\frac{1}{p} - \frac{1}{q} = \frac{s}{n}.$$

Then there exist constants $C(n,s,p), C(s,n) < \infty$ such that for all f in $\mathscr{S}(\mathbf{R}^n)$ we have

$$\big\|\mathcal{I}_s(f)\big\|_{L^q} \le C(n,s,p)\big\|f\big\|_{L^p}$$

and

$$\big\|\mathcal{I}_s(f)\big\|_{L^{\frac{n}{n-s},\infty}} \le C(n,s)\big\|f\big\|_{L^1}.$$

Consequently $\mathcal{I}_s$ has a unique extension on $L^p(\mathbf{R}^n)$ for all p with $1 \le p < \frac{n}{s}$ such that the preceding estimates are valid.

Proof. For a given nonnegative (and nonzero) function f in the Schwartz class we write

$$\int_{\mathbf{R}^n} f(x-y)|y|^{s-n}\,dy = I_1(f)(x) + I_2(f)(x),$$

where I_1 and I_2 are defined by

$$\begin{aligned} I_1(f)(x) &= \int_{|y|<R(x)} f(x-y)|y|^{s-n}\,dy, \\ I_2(f)(x) &= \int_{|y|\ge R(x)} f(x-y)|y|^{s-n}\,dy, \end{aligned}$$