Using this fact and (2.2.28), we conclude that

$$\Big\|\Big(\sum_{j=-M}^{M}|\Delta_j^{\Psi}(f)|^2\Big)^{\frac12}\Big\|_{L^p}\le C_{n,p,\Psi}\big\|f\big\|_{H^p}\,,$$

from which (2.2.25) follows directly by letting $M\to\infty$. We have now established (2.2.25) for $f\in H^p\cap L^1$. Using density, we can extend this estimate to all $f\in H^p$.

We now turn to the converse statement of the theorem. Assume that (2.2.26) holds for some tempered distribution f.

Set $\widehat{\eta}(\xi)=\widehat{\Psi}(\frac12\xi)+\widehat{\Psi}(\xi)+\widehat{\Psi}(2\xi)$. Then $\widehat{\eta}$ is supported in an annulus and is equal to 1 on the support of $\widehat{\Psi}$. Using Theorem 2.1.14 we obtain that for any $L\in\mathbf{Z}^+$ and $L'\in\mathbf{Z}^+\cup\{0\}$ with $L'<L$ the mapping

$$\{f_j\}_{L'\le|j|<L}\mapsto\sum_{L'\le|j|<L}\Delta_j^{\eta}(f_j)$$

maps $H^p(\mathbf{R}^n,\ell^2_{2L-2L'})$ to $H^p(\mathbf{R}^n)$; note that if $L'=0$, then $\ell^2_{2L-2L'}$ should be ℓ^2_{2L-1}. Indeed, Theorem 2.1.14 can be applied, since the family of kernels $\{\eta_{2^{-j}}\}_{L'\le|j|<L}$ satisfies $\sum_{L'\le|j|<L}|\partial_x^{\alpha}(\eta_{2^{-j}})(x)|\le C_\alpha\,|x|^{-n-|\alpha|}$, $x\neq0$, for all multiindices α and $\sum_{L'\le|j|<L}|\widehat{\eta_{2^{-j}}}|\le c'$ with constants independent of L,L'. Thus we have

$$\Big\|\sum_{L'\le|j|<L}\Delta_j^{\eta}(f_j)\Big\|_{H^p}\le C_{p,n,\Phi}\Big\|\sup_{t>0}\Big(\sum_{L'\le|j|<L}|\Phi_t*f_j|^2\Big)^{\frac12}\Big\|_{L^p}$$

when $\widehat{\Phi}$ is smooth, supported in $B(0,2)$, and $\widehat{\Phi}(0)\neq0$ and any $f_j\in H^p$. Taking[2] $f_j=\Delta_j^{\Psi}(f)$ and using that $\Delta_j^{\eta}\Delta_j^{\Psi}=\Delta_j^{\Psi}$, we deduce that for all $L\in\mathbf{Z}^+$ we have

$$\Big\|\sum_{L'\le|j|<L}\Delta_j^{\Psi}(f)\Big\|_{H^p}\le C_{p,n,\Phi}\Big\|\sup_{t>0}\Big(\sum_{L'\le|j|<L}|\Phi_t*\Delta_j^{\Psi}(f)|^2\Big)^{\frac12}\Big\|_{L^p}.$$

Applying Corollary 2.2.5 for some $r<p$ we arrive at the estimate

$$\begin{aligned}\Big\|\sum_{L'\le|j|<L}\Delta_j^{\Psi}(f)\Big\|_{H^p}&\le C_{p,n}\Big\|\Big(\sum_{L'\le|j|<L}|M(|\Delta_j^{\Psi}(f)|^r)|^{\frac2r}\Big)^{\frac12}\Big\|_{L^p}\\&=C_{p,n}\Big\|\Big(\sum_{L'\le|j|<L}|M(|\Delta_j^{\Psi}(f)|^r)|^{\frac2r}\Big)^{\frac r2}\Big\|_{L^{\frac pr}}^{\frac1r}.\end{aligned}$$

Since $r<\min(2,p)$, we use the $L^{p/r}(\mathbf{R}^n,\ell^{2/r}_{2L-2L'})$ to $L^{p/r}(\mathbf{R}^n,\ell^{2/r}_{2L-2L'})$ boundedness of the Hardy–Littlewood maximal operator (Theorem 5.6.6 in [156]) to obtain the inequality

[2] $f_j\in H^p$ since $\sup_{t>0}|\Phi_t*\Delta_j^{\Psi}(f)|\le C'M(|\Delta_j^{\Psi}(f)|^r)^{1/r}\in L^p$ for $r<p$ in view of (2.2.26).