for all $L,N>0$. We deduce the estimate

$$|V_j| \le C_{L,M,n,\Theta,\Omega} 2^{-|j|M}$$

for all M sufficiently large, which, in turn, yields the estimate

$$\sum_{j\in\mathbf{Z}} |V_j|^{\min(1,q)} < \infty .$$

We deduce from (2.2.20) that for all $x \in \mathbf{R}^n$ we have

$$\big\|\{2^{k\alpha} M^{**}_{b,k}(f;\Omega)(x)\}_k\big\|_{\ell^q} \le C_{\alpha,p,q,n,\Theta,\Omega} \big\|\{2^{k\alpha} M^{**}_{b,k}(f;\Theta)(x)\}_k\big\|_{\ell^q} . \tag{2.2.21}$$

Pick $r<\min(p,q)$. Estimate (2.2.7) in Lemma 2.2.3 with $b=\frac{n}{r}>\frac{n}{\min(p,q)}$ gives

$$2^{k\alpha} M^{**}_{b,k}(f;\Theta) \le C_2 2^{k\alpha} M(|\Delta_k^\Theta(f)|^r)^{\frac{1}{r}} = C_2 M(|2^{k\alpha}\Delta_k^\Theta(f)|^r)^{\frac{1}{r}} . \tag{2.2.22}$$

In view of (2.2.1) we have the identity

$$\Delta_k^\Theta = \Delta_k^\Theta \big(\Delta_{k-1}^\Psi + \Delta_k^\Psi + \Delta_{k+1}^\Psi\big) ,$$

and applying (2.2.14) to each term of the preceding sum yields

$$M(|2^{k\alpha}\Delta_k^\Theta(f)|^r)^{\frac{1}{r}} \le C' \Big(MM(|2^{k\alpha}\Delta_k^\Psi(f)|^r)\Big)^{\frac{1}{r}} . \tag{2.2.23}$$

Since $r<\min(p,q)$, we combine (2.2.21), (2.2.22), (2.2.23), and we use twice the $L^{p/r}(\mathbf{R}^n,\ell^{q/r})$ to $L^{p/r}(\mathbf{R}^n,\ell^{q/r})$ boundedness of the Hardy–Littlewood maximal operator (Theorem 5.6.6 in [156]) to complete the proof. □

2.2.4 The Littlewood–Paley Characterization of Hardy Spaces

We discuss an important characterization of Hardy spaces in terms of Littlewood–Paley square functions. The vector-valued Hardy spaces and the action of singular integrals on them are crucial tools in obtaining this characterization.

We have the following.

Theorem 2.2.9. *Let Ψ be a Schwartz function on $\mathbf{R}^n$ whose Fourier transform is nonnegative, supported in $\frac{6}{7} \le |\xi| \le 2$, equal to 1 on $1 \le |\xi| \le \frac{12}{7}$, and satisfies for all $\xi \ne 0$*

$$\sum_{j\in\mathbf{Z}} \widehat{\Psi}(2^{-j}\xi) = 1 . \tag{2.2.24}$$

Let Δ_j^Ψ be the Littlewood–Paley operators associated with Ψ and let $0<p\le 1$. Then there exists a constant $C=C_{n,p,\Psi}$ such that for all $f\in H^p(\mathbf{R}^n)$ we have

$$\Big\|\Big(\sum_{j\in\mathbf{Z}} |\Delta_j^\Psi(f)|^2\Big)^{\frac{1}{2}}\Big\|_{L^p} \le C\|f\|_{H^p} . \tag{2.2.25}$$