provided $-n < \mathrm{Re}\, z < 0$, in which case both $|\xi|^z$ and $|x|^{-z-n}$ are locally integrable functions. Dividing both sides of (1.2.2) by $\Gamma(\frac{n+z}{2})$ allows one to extend (1.2.2) to all complex numbers z as an identity between distributions; see Theorem 2.4.6 in [156] for details.

1.2.1 *Riesz Potentials*

When s is a positive real number, the operation $f \mapsto (-\Delta)^{-s/2} f$ is not really *differentiating* f; rather, it is *integrating*. For this reason, we introduce a slightly different notation that better reflects the nature of this operator.

Definition 1.2.1. Let s be a complex number with $0 < \mathrm{Re}\, s < \infty$. The *Riesz potential operator* of order s is

$$\mathcal{I}_s = (-\Delta)^{-s/2}.$$

Clearly $\mathcal{I}_s$ is well defined on Schwartz functions whose Fourier transform vanishes in a neighborhood of the origin; if $\mathrm{Re}\, s < n$, the function $\xi \mapsto |\xi|^{-s}$ is locally integrable, and thus $\mathcal{I}_s$ is well defined on the entire Schwartz class. Using identity (1.2.2), we express

$$\mathcal{I}_s(f)(x) = 2^{-s} \pi^{-\frac{n}{2}} \frac{\Gamma(\frac{n-s}{2})}{\Gamma(\frac{s}{2})} \int_{\mathbf{R}^n} f(x-y)|y|^{-n+s}\, dy,$$

and since this integral is convergent for all functions f in the Schwartz class, $\mathcal{I}_s$ is well defined on this space for all s with $\mathrm{Re}\, s > 0$.

We begin with a simple remark concerning the homogeneity of the operator $\mathcal{I}_s$.

Remark 1.2.2. Suppose that for some $s \in \mathbf{C}$, with $\mathrm{Re}\, s > 0$, we had an estimate

$$\big\|\mathcal{I}_s(f)\big\|_{L^q(\mathbf{R}^n)} \le C(p,q,n,s)\big\|f\big\|_{L^p(\mathbf{R}^n)} \tag{1.2.3}$$

for some positive indices p, q and all $f \in \mathscr{S}(\mathbf{R}^n)$. Then p and q must be related by

$$\frac{1}{p} - \frac{1}{q} = \frac{\mathrm{Re}\, s}{n}. \tag{1.2.4}$$

This follows by applying (1.2.3) to the dilation $\delta^\lambda(f)(x) = f(\lambda x)$, $\lambda > 0$, in lieu of f. Indeed, replacing f by $\delta^\lambda(f)$ in (1.2.3) and using the identity

$$\mathcal{I}_s(\delta^\lambda(f)) = \lambda^{-s} \delta^\lambda(\mathcal{I}_s(f))$$

which follows by a changes of variables, we obtain

$$\lambda^{-\frac{n}{q} - \mathrm{Re}\, s} \big\|\mathcal{I}_s(f)\big\|_{L^q(\mathbf{R}^n)} \le C(p,q,n,s) \lambda^{-\frac{n}{p}} \big\|f\big\|_{L^p(\mathbf{R}^n)} \tag{1.2.5}$$