argument proves this assertion. So we prove that $w \mapsto \int_{\mathbf{R}^n} |x|^w \psi(x)\,dx$ is analytic. We fix w_0 with $\operatorname{Re} w_0 > -n$ and pick $\delta > 0$ such that $\operatorname{Re} w_0 - 2\delta > -n$. Then

$$\lim_{w\to 0} \frac{1}{w}\left[\int_{\mathbf{R}^n} |x|^{w+w_0}\psi(x)\,dx - \int_{\mathbf{R}^n} |x|^{w_0}\psi(x)\,dx\right] = \int_{\mathbf{R}^n} |x|^{w_0}\big(\log|x|\big)\psi(x)\,dx,$$

since $\lim_{w\to 0} \frac{|x|^w - 1}{w} = \log|x|$. The passing of the limit inside the integral is justified from the LDCT via the inequality in (2.7.8), which holds for $0 < |w| < \delta$, combined with the fact that $|x|^{w_0}\max(|x|^{2\delta}, |x|^{-2\delta})\psi(x)\log|x|$ is integrable over $\mathbf{R}^n$. $\square$

It turns out that the function in (2.7.7) is entire and thus u_z extends to an entire-valued tempered distribution whose Fourier transform is u_{-z-n}. On this see [31].

Exercises

2.7.1. Let $\Phi \in \mathscr{S}(\mathbf{R}^n)$ with $\int_{\mathbf{R}^n} \Phi(x)\,dx = 1$ and for $\varepsilon > 0$ let $\Phi_\varepsilon(x) = \varepsilon^{-n}\Phi(\varepsilon^{-1}x)$. Show that $\Phi_\varepsilon \to \delta_0$ in $\mathscr{S}'(\mathbf{R}^n)$ and that $\Phi_\varepsilon * f \to f$ in $\mathscr{S}$ for every $f \in \mathscr{S}(\mathbf{R}^n)$. Conclude that $\Phi_\varepsilon * u \to u$ in $\mathscr{S}'$ for every $u \in \mathscr{S}'(\mathbf{R}^n)$.

2.7.2. For $\varphi \in \mathscr{S}(\mathbf{R}^n)$ prove that $(\tau^{-he_j}\varphi - \varphi)/h \to \partial_j\varphi$ in $\mathscr{S}$ as $h \to 0$.

2.7.3. On the real line consider the tempered distribution u_z of Example 2.7.4. Use the Taylor expansion at the origin of a function $\varphi \in \mathscr{S}(\mathbf{R})$

$$\varphi(x) = \sum_{k=0}^{N} \frac{\varphi^{(k)}(0)}{k!}x^k + \frac{x^{N+1}}{N!}\int_0^1 (1-t)^N \varphi^{(N+1)}(tx)dt$$

for an arbitrary even positive integer N, to write

$$\int_{|x|<1} |x|^z\varphi(x)dx = \sum_{\substack{k=0\\ k \text{ even}}}^{N} \frac{\varphi^{(k)}(0)}{k!}\frac{\omega_{n-1}}{z+k+1} + \int_{|x|<1}\int_0^1 \frac{\varphi^{(N+1)}(tx)}{N!}(1-t)^N dt|x|^z x^{N+1}dx.$$

Deduce the analyticity of the function $z \mapsto \int_{|x|<1} |x|^z\varphi(x)\,dx$ on $\mathbf{C}\setminus E$, where $E = \{-1,-3,-5,\dots\}$. Conclude from this that the function $z \mapsto \langle u_z, \varphi\rangle$ is entire.
$\big[$*Hint:* The function $z \mapsto \Gamma(\frac{z+1}{2})^{-1}$ has zeros of order 1 at $-1,-3,-5,\dots.\big]$

2.7.4. Suppose that f is a tempered distribution on $\mathbf{R}^n$ whose Fourier transform coincides with an integrable and compactly supported function. Prove that $f \in \mathscr{C}^\infty$.
$\big[$*Hint:* Show first that f can be identified with the function $x \mapsto \int_{\mathbf{R}^n} \widehat{f}(\xi)e^{2\pi i x\cdot\xi}d\xi.\big]$

2.7.5. Let $a > 0$. Using that the Fourier transform of $e^{-\pi|x|^2}$ is itself, show that
(a) The Fourier transform of $e^{-\pi a|x|^2}$ on $\mathbf{R}^n$ is $a^{-n/2}e^{-\pi|x|^2/a}$.
(b) The Fourier transform of $e^{-\pi(a+it)|x|^2}$ is $(a+it)^{-n/2}e^{-\pi|x|^2/(a+it)}$, $t \in \mathbf{R}$.
(c) The distributional Fourier transform of $e^{-i\pi t|x|^2}$ is $(it)^{-n/2}e^{i\pi|x|^2/t}$, $t \in \mathbf{R}\setminus\{0\}$.