$$
\begin{aligned}
|\partial^\alpha(\varphi * u)(x)| &\le C \sum_{\substack{|\gamma|\le M\\ |\beta|\le K}} \sup_{y\in\mathbf{R}^n} |y^\gamma \partial^{\alpha+\beta}(\tau^x\widetilde{\varphi})(y)| \\
&= C \sum_{\substack{|\gamma|\le M\\ |\beta|\le K}} \sup_{y\in\mathbf{R}^n} |(x+y)^\gamma (\partial^{\alpha+\beta}\widetilde{\varphi})(y)| \\
&\le (1+|x|)^M \Big[C M^n \sum_{|\beta|\le K} \sup_{y\in\mathbf{R}^n} (1+|y|)^M |(\partial^{\alpha+\beta}\widetilde{\varphi})(y)| \Big],
\end{aligned}
$$

and this yields (2.7.1), with $C_{\alpha,u,\varphi}$ being the expression in the square brackets. □

Lemma 2.7.2. *The Riemann sums of the integral in (2.7.2) converge to this integral in the topology of $\mathscr{S}$.*

Proof. For each $N \in \mathbf{Z}^+$ we partition $[-N,N]^n$ into a union of $(2N^2)^n$ cubes Q_j of side length $1/N$ and we let y_j be the center of each Q_j. We will show that for multi-indices α,β the following Riemann sum minus the corresponding integral

$$
D_N(x) = x^\alpha \Bigg[\sum_{j=1}^{(2N^2)^n} \psi(y_j)\partial_x^\beta \widetilde{\varphi}(x-y_j)|Q_j| - \int_{\mathbf{R}^n} \psi(y)\partial_x^\beta \widetilde{\varphi}(x-y)\,dy \Bigg] \tag{2.7.3}
$$

converges to zero in $L^\infty(\mathbf{R}^n)$ as $N\to\infty$. We write

$$
\begin{aligned}
& x^\alpha \Bigg[\sum_{j=1}^{(2N^2)^n} \psi(y_j)\partial_x^\beta \widetilde{\varphi}(x-y_j)|Q_j| - \sum_{j=1}^{(2N^2)^n} \int_{Q_j} \psi(y)\partial_x^\beta \widetilde{\varphi}(x-y)\,dy \Bigg] \\
&= x^\alpha \sum_{j=1}^{(2N^2)^n} \int_{Q_j} \Big[\psi(y_j)\partial_x^\beta \widetilde{\varphi}(x-y_j) - \psi(y)\partial_x^\beta \widetilde{\varphi}(x-y) \Big]\,dy \\
&= x^\alpha \sum_{j=1}^{(2N^2)^n} \int_{Q_j}\int_0^1 \nabla\big[\psi\partial_x^\beta\widetilde{\varphi}(x-\cdot)\big]\big((1-\theta)y+\theta y_j\big)\cdot(y_j-y)\,d\theta\,dy \qquad (2.7.4)
\end{aligned}
$$

by the mean value theorem. Using estimates for Schwartz functions and the simple inequality $|\nabla(FG)| \le \sum_{k=1}^n (|\partial_k F|\,|G| + |F|\,|\partial_k G|)$, for $y\in Q_j$ we estimate

$$
\big|x^\alpha \nabla\big[\partial_x^\beta \psi\widetilde{\varphi}(x-\cdot)\big](\xi)\cdot(y_j-y)\big| \le \frac{C_M\,|x|^{|\alpha|}}{(1+|x-\xi|)^{M/2}}\frac{1}{(2+|\xi|)^M}\frac{\sqrt{n}}{2N}
$$

when $M > 2|\alpha|+2n$, where $\xi = (1-\theta)y+\theta y_j$. The last expression is bounded by

$$
\frac{C_M\,|x|^{|\alpha|}}{(1+|x|)^{M/2}}\frac{1}{(2+|\xi|)^{M/2}}\frac{\sqrt{n}}{2N} \le \frac{C_M\,|x|^{|\alpha|}}{(1+|x|)^{M/2}}\frac{1}{(1+|y|)^{M/2}}\frac{\sqrt{n}}{2N},
$$

since $|\xi| \ge |y| - \theta|y-y_j| \ge |y| - \frac{\sqrt{n}}{2N} \ge |y|-1$ for $N\ge\sqrt{n}$. Inserting this estimate in (2.7.4) and using (2.7.3) and the fact that $\mathbf{R}^n = \cup_j Q_j \cup ([-N,N]^n)^c$, we obtain