One can check that the operations of translation, dilation, reflection, and differentiation are continuous on tempered distributions.

Example 2.6.14. Let $x_0 \in \mathbf{R}^n$. Then we have $\widetilde{\delta_{x_0}} = \delta_{-x_0}$ (in particular, $\widetilde{\delta_0} = \delta_0$), also $(\delta_0)^t = t^{-n}\delta_0$, and $\tau^{x_0}\delta_0 = \delta_{x_0}$.

We now define the product of a function and a distribution.

Definition 2.6.15. Let $u \in \mathscr{S}'$ and let h be a $\mathscr{C}^\infty$ tempered function whose derivatives are also tempered. This means that for all multi-indices γ there are $C_\gamma, k_\gamma > 0$ such that $|\partial^\gamma h(x)| \le C_\gamma (1+|x|)^{k_\gamma}$. We define the product of h and u by setting

$$\langle hu, \varphi \rangle = \langle u, h\varphi \rangle, \qquad \varphi \in \mathscr{S}. \tag{2.6.13}$$

To verify that hu is a well-defined element of $\mathscr{S}'$, we first verify that $h\varphi$ lies in $\mathscr{S}$; indeed, for each pair of multi-indices α, β we have

$$\rho_{\alpha,\beta}(h\varphi) \le \sum_{\gamma \le \beta} C_\gamma C_{n,k_\gamma}^{-1} \binom{\beta_1}{\gamma_1} \cdots \binom{\beta_n}{\gamma_n} \sum_{|\delta| \le k_\gamma} \rho_{\alpha+\delta, \beta-\gamma}(\varphi) < \infty,$$

in view of Leibniz's rule, where C_{n,k_γ} are the constants in (1.7.3). This implies that $|\langle hu, \varphi \rangle|$ is bounded by a finite sum of $\rho_{\gamma,\delta}(\varphi)$, thus hu lies in $\mathscr{S}'(\mathbf{R}^n)$.

To define the convolution of a function with a tempered distribution, we examine an identity for functions. Observe that for φ, ψ in $\mathscr{S}(\mathbf{R}^n)$ and any integrable function[6] g on $\mathbf{R}^n$ the identity holds:

$$\int_{\mathbf{R}^n} (\varphi * g)(x)\psi(x)\,dx = \int_{\mathbf{R}^n} g(x)(\widetilde{\varphi} * \psi)(x)\,dx. \tag{2.6.14}$$

Motivated by (2.6.14), we give the following definition:

Definition 2.6.16. Let $u \in \mathscr{S}'$ and $\varphi \in \mathscr{S}$. Define the *convolution* $\varphi * u$ as follows:

$$\langle \varphi * u, \psi \rangle = \langle u, \widetilde{\varphi} * \psi \rangle, \qquad \psi \in \mathscr{S}(\mathbf{R}^n). \tag{2.6.15}$$

We note that $\varphi * u$ lies in $\mathscr{S}'(\mathbf{R}^n)$, since for all multi-indices α, β we have

$$\begin{aligned}
\rho_{\alpha,\beta}(\widetilde{\varphi} * \psi) &\le \sup_{x \in \mathbf{R}^n} \int_{\mathbf{R}^n} |x|^{|\alpha|} |\varphi(y-x)|\,|\partial^\beta \psi(y)|\,dy \\
&\le 2^{|\alpha|} \sup_{x \in \mathbf{R}^n} \int_{\mathbf{R}^n} (|y-x|^{|\alpha|} + |y|^{|\alpha|}) |\varphi(y-x)|\,|\partial^\beta \psi(y)|\,dy \\
&\le C_{\alpha,\beta,\phi} \big(\rho_{0,\beta}(\psi) + \sum_{|\gamma|=|\alpha|} \rho_{\gamma,\beta}(\psi) \big),
\end{aligned}$$

using the inequality $|x|^{|\alpha|} \le 2^{|\alpha|}|x-y|^{|\alpha|} + 2^{|\alpha|}|y|^{|\alpha|}$ and (1.7.2).

[6] In fact, any locally integrable function that is tempered at infinity.