$\big[$*Hint:* Let $K(x) = (1+|x|)^{-n-\gamma}$. Show that condition (2.5.7) is valid for all $x \in \mathbf{R}^n$.$\big]$

2.5.3. Show that conditions on K in Theorem 2.5.1 can be relaxed as follows:
(a) $|K(x)| \le L(|x|)$ for some decreasing function $L : (0,\infty) \to [0,\infty)$.
(b) $L(|x|)$ lies in $L^1(\mathbf{R}^n)$.
$\big[$*Hint:* Use that $(1-2^{-n})v_n \sum_{k\in\mathbf{Z}} 2^{(k+1)n} L(2^k) \le 2^n \int_{\mathbf{R}^n} L(|x|)\,dx.$$\big]$

2.5.4. Under the hypotheses of Theorem 2.5.7, if additionally f lies in $L^\infty(\mathbf{R}^n)$ and is continuous on a closed ball $\overline{B(x_0,\delta_0)}$ on $\mathbf{R}^n$, prove that

$$(K_t * f)(x) \to c f(x_0) \qquad \text{as } (x,t) \to (x_0, 0^+).$$

2.5.5. (Borel–Cantelli lemma) Suppose that $\{f_t\}_{t>0}$ is a family of measurable functions on a compact subset K of $\mathbf{R}^n$ (or on any measure space with finite measure). Suppose that for any $\varepsilon > 0$ the sets $A_t(\varepsilon) = \{x \in K : \ |f_t(x)| \ge \varepsilon\}$ satisfy

$$\sum_{k=1}^{\infty} |A_{t_k}(\varepsilon)| < \infty$$

for any sequence $t_k > 0$ that tends to zero. Prove that $f_t \to 0$ a.e. as $t \to 0^+$.
$\big[$*Hint:* Show first that for any sequence $t_k \to 0^+$ we have $\big|\bigcap_{m=1}^\infty \bigcup_{k=m}^\infty A_{t_k}(\varepsilon)\big| = 0.$$\big]$

2.6 Tempered Distributions

An integrable function g is almost everywhere uniquely determined[4] by the integrals $\int_{\mathbf{R}^n} g\,\varphi\,dx$, where φ ranges over $\mathscr{C}_0^\infty(\mathbf{R}^n)$. For this reason we can identify g by the functional $L_g(\varphi) = \int_{\mathbf{R}^n} g\,\varphi\,dx$, acting on $\mathscr{C}_0^\infty(\mathbf{R}^n)$. Functionals acting on nice classes of functions are called *generalized functions* or *distributions*. Viewing functions as functionals allows us to perform operations to them that would normally not be possible. For instance, one can define the partial derivative of a function $g \in L^1(\mathbf{R}^n)$ to be the functional $\partial_1 L_g$ given by $\partial_1 L_g(\varphi) = -L_g(\partial_1 \varphi)$ for all $\varphi \in \mathscr{C}_0^\infty(\mathbf{R}^n)$. For such reasons, the theory of distributions provides not only a mathematically sound but also a flexible framework to work with. The theory of distributions is vast and extensive, but here we focus only on some basic facts concerning tempered distributions.

A *linear functional* ~~on~~ u on the space of Schwartz functions $\mathscr{S}(\mathbf{R}^n)$ is a linear mapping from $\mathscr{S}(\mathbf{R}^n)$ to the complex numbers. The action $u(\varphi)$ of u on a Schwartz function φ is denoted by $\langle u, \varphi \rangle$. Recall that for $\varphi \in \mathscr{S}(\mathbf{R}^n)$ and multi-indices α, β the expressions

$$\rho_{\alpha,\beta}(\varphi) = \sup_{x\in\mathbf{R}^n} |x^\alpha \partial^\beta \varphi(x)| \tag{2.6.1}$$

are called *Schwartz seminorms* of φ.

[4] Exercise 1.9.7