Remark 2.5.8. Theorem 2.5.7 could have been stated in the following form: (2.5.9) is valid whenever both (2.5.7) and (2.5.8) hold at a point $x \in \mathscr{L}_f$.

Remark 2.5.9. Let K be as in Theorem 2.5.7. If K has compact support, then condition (2.5.7) holds for any locally integrable function f. Indeed, if K is supported in a ball $B(0,M)$, then the integral in (2.5.7) is over the set $\theta \le |y| \le Mt$ and this set becomes empty when $t < \theta/M$, so the integral is zero for t sufficiently small.

We also observe that condition (2.5.7) can be derived from

$$\int_{\mathbf{R}^n} \frac{|f(z)|}{(1+|z|)^{n+\gamma}}\, dz < \infty. \tag{2.5.15}$$

Indeed, assuming (2.5.15), for any $x \in \mathbf{R}^n$, we obtain

$$\int_{\mathbf{R}^n} \frac{|f(x-y)|}{(1+|y|)^{n+\gamma}}\, dy = \int_{\mathbf{R}^n} \frac{|f(z)|}{(1+|x-z|)^{n+\gamma}}\, dz < \infty \tag{2.5.16}$$

by splitting the z integral in (2.5.16) in the regions $|z| \le 2|x|$ and $|z| \ge 2|x|$; in the latter case $|z| \approx |z-x|$ so (2.5.15) applies. Also the integral over the region $|z| \le 2|x|$ is finite as f is locally integrable. Then for $|y| \ge \theta$ and $t > 0$ we have

$$|K_t(y)| \le \frac{A}{t^n}\left|\frac{y}{t}\right|^{-n}\left|\frac{y}{t}\right|^{-\gamma} = A\frac{t^\gamma}{|y|^{n+\gamma}} \le A\,t^\gamma\Big(\frac{\theta+1}{\theta}\Big)^{n+\gamma}\frac{1}{(1+|y|)^{n+\gamma}}\,.$$

Combining this estimate with (2.5.16), we deduce (2.5.7).

Example 2.5.10. Let $A > 0$, $0 < \gamma < n$ and $|K(x)| \le A\,|x|^{-n}\min(|x|^\gamma, |x|^{-\gamma})$ when $x \ne 0$. Then Theorem 2.5.7 applies in the following situations:

(a) $f \in L^p(\mathbf{R}^n)$, $1 \le p \le \infty$.

(b) $|f(x)| \le C(1+|x|)^\tau$ for $\tau < \gamma$.

(c) $|f(x)| \le C(1+|x|)^\tau$ for $\tau < 1$ and K is the Poisson kernel P.

(d) $|f(x)| \le Ce^{|x|^\delta}$ for $0 \le \delta < 2$ and $K(x) = e^{-\pi|x|^2}$.

(e) $f \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$ and K has compact support.

Exercises

2.5.1. Verify that in the five cases of Example 2.5.10, condition (2.5.7) is satisfied.

2.5.2. Let $0 < \gamma < n$ and let x_0 be a Lebesgue point of a function f in $L^q(\mathbf{R}^n)$ where $\frac{n}{\gamma} < q \le \infty$. Prove that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^n}\int_{\mathbf{R}^n}\frac{f(x)-f(x_0)}{\big(1+\frac{|x-x_0|}{\varepsilon}\big)^{n+\gamma}}\,dx = 0.$$