$$
\begin{aligned}
&= \int_0^{\delta_0} \frac{d}{dr}\Big[\int_0^r F(\rho)\,d\rho\Big]\frac{1}{t^n}L\Big(\frac{r}{t}\Big)\,dr \\
&= \bigg(\frac{1}{\delta_0^n}\int_0^{\delta_0} F(\rho)\,d\rho\bigg)\frac{\delta_0^n}{t^n}L\Big(\frac{\delta_0}{t}\Big) - \int_0^{\delta_0}\bigg(\frac{1}{r^n}\int_0^r F(\rho)\,d\rho\bigg)\frac{r^n}{t^n}\frac{1}{t}L'\Big(\frac{r}{t}\Big)dr \\
&= Q_f(x_0),
\end{aligned}
$$

having used (2.5.13) with $\phi(r) = \int_0^r F(\rho)\,d\rho$ and $b = \delta_0$. Since we picked $\delta_0 < 1$ it follows that for any $t > 0$

$$
\int_0^{\delta_0} L\Big(\frac{r}{t}\Big)|\phi'(r)|dr = \int_{|y|<\delta_0} |f(x_0 - y) - f(x_0)|\,L\Big(\frac{|y|}{t}\Big)\,dy < \infty
$$

so (2.5.14) is valid and thus (2.5.13) is justified. Next we use (2.5.10) and the fact $-L' > 0$ to obtain the estimate

$$
\begin{aligned}
Q_f(x_0) &\le \frac{\gamma v_n \varepsilon}{4\omega_{n-1}A}\bigg[\frac{\delta_0^n}{t^n}L\Big(\frac{\delta_0}{t}\Big) - \int_0^{\delta_0}\frac{r^n}{t^n}\frac{1}{t}L'\Big(\frac{r}{t}\Big)\,dr\bigg] \\
&= \frac{\gamma v_n \varepsilon}{4\omega_{n-1}A}\bigg[\frac{\delta_0^n}{t^n}L\Big(\frac{\delta_0}{t}\Big) - \int_0^{\delta_0/t} r^n L'(r)\,dr\bigg] \\
&= \frac{\gamma v_n \varepsilon}{4\omega_{n-1}A}\,n\bigg[\int_0^{\delta_0/t} r^{n-1}L(r)\,dr\bigg] \\
&\le \frac{\gamma \varepsilon}{4\omega_{n-1}A}\,n\,v_n A\bigg[\int_0^{\infty}\min(r^{\gamma}, r^{-\gamma})\,\frac{dr}{r}\bigg] \\
&= \frac{\varepsilon}{2},
\end{aligned}
$$

where the second equality is based on (2.5.13) with $\phi(r) = r^n$. Then for $0 < t < \delta$, combining the estimates derived for the two terms in (2.5.11), we deduce

$$
|(K_t * f)(x_0) - c f(x_0)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
$$

and this proves (2.5.9).

Finally we show that (2.5.8) is satisfied for almost all $x \in \mathscr{L}_f$, and thus the claimed almost convergence is valid. For every $N \in \mathbf{Z}^+$ we have

$$
\int_{|x|<N}\bigg[\int_{|y|\le 1}|f(x-y)|\,\frac{dy}{|y|^{n-\gamma}}\bigg]dx \le \bigg(\int_{|y|\le 1}\frac{dy}{|y|^{n-\gamma}}\bigg)\int_{|x'|\le N+1}|f(x')|\,dx' < \infty.
$$

Consequently the integral inside the square brackets is finite for almost all points x in the ball $B(0,N)$, so letting $N \to \infty$ through the positive integers we obtain (2.5.8) for almost all points x in $\mathbf{R}^n$. □

We note that there is no restriction in assuming that $\gamma < n$ as the size estimate on K deteriorates as γ decreases to 0.