Finite linear combinations of characteristic functions of cubes with sides parallel to the axes on $\mathbf{R}^n$ are called *step functions*. It is known that step functions are dense in $L^p(\mathbf{R}^n)$ for every $p<\infty$.

A natural measure on the space $\mathbf{Z}$ is *counting measure* ν defined on subsets A of $\mathbf{Z}$ by $\nu(A) =$ cardinality of A. For $0<p\le\infty$, $\ell^p(\mathbf{Z})$ denotes the space $L^p(\mathbf{Z},\nu)$.

Example 1.1.13. The spaces $L^p([0,1])$ decrease and the spaces $\ell^p(\mathbf{Z})$ increase as p increases. Indeed for $0<p<q\le\infty$, by Hölder's inequality, we have

$$\big\|f\big\|_{L^p([0,1])} \le |[0,1]|^{\frac{1}{p}-\frac{1}{q}}\big\|f\big\|_{L^q([0,1])} = \big\|f\big\|_{L^q([0,1])},$$

so $L^q([0,1])$ is contained in $L^p([0,1])$.

Now $\big\|\{b(k)\}_k\big\|_{\ell^p(\mathbf{Z})} = \big(\int_{\mathbf{Z}}|b(k)|^p d\nu\big)^{\frac{1}{p}} = \big(\sum_{k\in\mathbf{Z}}|b(k)|^p\big)^{\frac{1}{p}}$; hence

$$\big\|\{b(k)\}_k\big\|_{\ell^q} = \Big(\sum_{k\in\mathbf{Z}}|b(k)|^{q-p+p}\Big)^{\frac{1}{q}} \le \big\|\{b(k)\}_k\big\|_{\ell^\infty}^{\frac{q-p}{q}}\big\|\{b(k)\}_k\big\|_{\ell^p}^{\frac{p}{q}} \le \big\|\{b(k)\}_k\big\|_{\ell^p}$$

as clearly $\big\|\{b(k)\}_k\big\|_{\ell^\infty} \le \big\|\{b(k)\}_k\big\|_{\ell^p(\mathbf{Z})}$. This shows that $\ell^p(\mathbf{Z})$ embeds in $\ell^q(\mathbf{Z})$.

Exercises

1.1.1. Let $\Phi:[0,\infty)\to[0,\infty)$ be a continuous increasing function with $\Phi(0)=0$ such that for some $K>0$ and all $t,s\in[0,\infty)$ we have $\Phi(t+s)\le K(\Phi(t)+\Phi(s))$ (*quasi-subadditivity*). Let f, f_n, $n=1,2,\dots$, be measurable functions on (X,μ) that satisfy $f_n\to f$ a.e. and $\int_X\Phi(|f|)\,d\mu<\infty$. Prove that

$$\int_X\Phi(|f_n-f|)\,d\mu\to 0 \iff \int_X\Phi(|f_n|)\,d\mu\to\int_X\Phi(|f|)\,d\mu$$

as $n\to\infty$. Apply this result to $\Phi(t)=|t|^p$ and $\Phi(t)=|t|^p\ln(|t|+1)^q$, $0<p,q<\infty$. $\big[$*Hint:* Apply Theorem 1.1.8 (b) with $g_n = K\big(\Phi(|f_n|)+\Phi(|f|)\big)$.$\big]$

1.1.2. Suppose that $1\le p<\infty$ and f is a measurable function on a σ-finite measure space (X,μ) such that fg lies in L^1 for any $g\in L^{p'}$. Prove that $f\in L^p$. Derive the same conclusion if $p=\infty$. $\big[$*Hint:* Show by contradiction that there is a positive finite constant C such that $\|fg\|_{L^1}\le C\|g\|_{L^{p'}}$ for all $g\in L^{p'}$. Then use Proposition 1.1.4.$\big]$

1.1.3. Let $0<p_0\le\infty$. Show that the function $h(t)=|t|^{-\frac{1}{p_0}}\big(1+\big|\log|t|\big|\big)^{-\frac{1}{p_0}-1}$ lies in $L^p(\mathbf{R})$ if and only if $p=p_0$.

1.1.4. Let f_j, $1\le j\le n$, be real-valued measurable functions on a measure space (X,μ) and suppose that

$$\big\|f_1\big\|_{L^p} = \big\|f_2\big\|_{L^p} = \cdots = \big\|f_n\big\|_{L^p} = \Big\|\frac{f_1+\cdots+f_n}{n}\Big\|_{L^p}$$