and we bound the absolute value of the last integral by

$$\int_{|y|<\delta_0} |f(x_0-y)-f(x_0)|\,|K_t(y)|\,dy+\int_{|y|\geq\delta_0} |f(x_0-y)-f(x_0)|\,|K_t(y)|\,dy. \quad (2.5.11)$$

We first estimate the second integral in (2.5.11). For $t>0$ we have

$$\begin{aligned}&\int_{|y|\geq\delta_0} |f(x_0-y)-f(x_0)|\,|K_t(y)|\,dy\\ &\qquad\leq \int_{|y|\geq\delta_0} |f(x_0-y)|\,|K_t(y)|\,dy+|f(x_0)|\,A\,t^\gamma\int_{|y|\geq\delta_0} |y|^{-n-\gamma}\,dy.\end{aligned}$$

We pick $\delta>0$ such that the sum above is smaller than $\varepsilon/2$ when $0<t<\delta$, in view of (2.5.7) and the appearance of t^γ. Note that δ depends on f, x_0, n, γ, and δ_0.

To handle the first integral in (2.5.11) we use polar coordinates to write

$$\begin{aligned}\frac{1}{r^n}\int_{|y|<r}|f(x_0-y)-f(x_0)|dy &= \frac{1}{r^n}\int_0^r \rho^{n-1}\int_{\mathbf{S}^{n-1}}|f(x_0-\rho\theta)-f(x_0)|\,d\theta\,d\rho\\ &= \frac{1}{r^n}\int_0^r F(\rho)\,d\rho, \qquad (2.5.12)\end{aligned}$$

where

$$F(\rho)=\rho^{n-1}\int_{\mathbf{S}^{n-1}}|f(x_0-\rho\theta)-f(x_0)|\,d\theta, \qquad \rho>0.$$

Since $|f-f(x_0)|$ is integrable over any ball centered at x_0, it follows that $F(\rho)$ is defined for almost all $\rho>0$. In view of (2.5.10), the expression in (2.5.12) is at most $\frac{\gamma v_n\varepsilon}{4\omega_{n-1}A}$ when $0<r\leq\delta_0$. Now set $L(r)=A\,r^{-n}\min(r^\gamma,r^{-\gamma})$ defined for $r>0$. This function is continuous on $(0,\infty)$ and continuously differentiable on $(0,1)\cup(1,\infty)$. Also the integration-by-parts identity

$$\int_0^b L\Big(\frac{r}{t}\Big)\phi'(r)dr = L\Big(\frac{b}{t}\Big)\phi(b)-\int_0^b \frac{1}{t}L'\Big(\frac{r}{t}\Big)\phi(r)dr \qquad (2.5.13)$$

is valid for all $t>0$, whenever ϕ is a differentiable function on $(0,b)$ satisfying

$$0\leq\phi(r)\leq Cr^n \qquad \text{and} \qquad \int_0^b L\Big(\frac{r}{t}\Big)|\phi'(r)|dr<\infty. \qquad (2.5.14)$$

If $b>1$ this can be seen by splitting the interval of integration in $(0,1)$ and $(1,b)$ and summing the outputs using that $\lim_{\delta\to 0}L(\delta/t)\phi(\delta)=0$. Since $\gamma<n$ we have $L'<0$ on $(0,1)\cup(1,\infty)$ and L' is undefined at 1. Now for any $t>0$ we write

$$\begin{aligned}&\int_{|y|<\delta_0}|f(x_0-y)-f(x_0)|\,|K_t(y)|\,dy\\ &\qquad\leq\int_{|y|<\delta_0}|f(x_0-y)-f(x_0)|\,\frac{1}{t^n}L\Big(\frac{|y|}{t}\Big)\,dy\end{aligned}$$