Essentially the same proof yields the same more general result.

Theorem 2.5.6. *Fix $1 \le p < \infty$. Let $\{T_t\}_{t>0}$ and T be linear operators defined on $L^p(\mathbf{R}^n)$ such that $T_t(\varphi)(x) \to T(\varphi)(x)$ as $t \to 0$ for all $\varphi \in \mathscr{C}_0^\infty(\mathbf{R}^n)$ and all $x \in \mathbf{R}^n$. Suppose that $T^{(*)}(f) = \sup_{t>0} |T_t(f)|$ is a bounded operator from $L^p(\mathbf{R}^n)$ to $L^{p,\infty}(\mathbf{R}^n)$. Then for all $f \in L^p(\mathbf{R}^n)$, $T_t(f) \to T(f)$ a.e. as $t \to 0$.*

Proof. Adapt the proof of Theorem 2.5.5 replacing $K_t * f$ by $T_t(f)$ and $\mathcal{M}$ by $T^{(*)}$. □

Theorem 2.5.5 does not cover the case of $p = \infty$, in view of the lack of a nice dense subspace of L^∞. A different proof of Theorem 2.5.5 can be given that not only covers the case $p = \infty$, but also allows the function f to have moderate growth at infinity, or even be locally integrable, if K has compact support. But the most important ingredient of this proof is that it relates the set of almost everywhere convergence to the Lebesgue set $\mathscr{L}_f$ of f.

Theorem 2.5.7. *Let $K \in L^1(\mathbf{R}^n)$ satisfy $|K(x)| \le A\,|x|^{-n} \min(|x|^\gamma, |x|^{-\gamma})$ when $x \neq 0$, where $A > 0$ and $0 < \gamma < n$. Let $f \in L^1_{\text{loc}}(\mathbf{R}^n)$. Suppose that*

$$\lim_{t\to 0^+} \int_{|y|\ge\theta} |f(x-y)||K_t(y)|\,dy = 0 \qquad \text{for all } \theta > 0 \text{ and } x \in \mathbf{R}^n. \tag{2.5.7}$$

Then for every $x \in \mathscr{L}_f$ for which

$$\int_{|y|\le 1} |f(x-y)|\,|y|^{-n+\gamma}\,dy < \infty \tag{2.5.8}$$

we have

$$\lim_{t\to 0^+} (K_t * f)(x) = c f(x), \tag{2.5.9}$$

where $c = \int_{\mathbf{R}^n} K(y)\,dy$. Consequently, $K_t * f \to cf$ a.e. as $t \to 0^+$.

Proof. We fix f and K as in the statement of the theorem and $x_0 \in \mathscr{L}_f$ such that (2.5.8) is satisfied. We begin with the observation that (2.5.7) with $\theta = 1$ and (2.5.8), combined with the fact that $|K_t(y)| \le t^{-n}|y/t|^{-n+\gamma}$, yield

$$(|f| * |K_t|)(x_0) < \infty \qquad \text{for } t \text{ sufficiently small depending on } x_0.$$

We will prove (2.5.9) for $x = x_0$. Let $\varepsilon > 0$ be given. As $x_0 \in \mathscr{L}_f$ there is a $\delta_0 > 0$ (which we pick to satisfy $\delta_0 < 1$) such that

$$0 < r \le \delta_0 \implies \frac{1}{v_n r^n} \int_{|y|<r} |f(x_0-y) - f(x_0)|\,dy < \frac{\gamma}{4\omega_{n-1}A}\varepsilon. \tag{2.5.10}$$

Here $v_n = |B(0,1)|$ and $\omega_{n-1} = |\mathbf{S}^{n-1}|$. Since $\int_{\mathbf{R}^n} K_t(y)\,dy = c$ for any $t > 0$, we write

$$(K_t * f)(x_0) - cf(x_0) = \int_{\mathbf{R}^n} K_t(y)\big(f(x_0-y) - f(x_0)\big)\,dy,$$