Proof. Use Theorem 2.5.1 with $K(x) = (1+|x|)^{-n-\gamma}$. □

The conditions on K in Theorem 2.5.1 are weakened in Exercise 2.5.3. Another proof of Theorem 2.5.1 can be given which explicitly relates the value of constant $C_{n,\gamma}$ in (2.5.1) to K.

Proposition 2.5.3. *Let $K(x)$ be a nonnegative integrable function on $\mathbf{R}^n$, which is radial and decreasing*[3] *on $[0,\infty)$ as a function of $|x|$. Then for $f \in L^1_{\text{loc}}(\mathbf{R}^n)$ and any $x \in \mathbf{R}^n$ we have*

$$\sup_{t>0}(K_t * |f|)(x) \le \big\|K\big\|_{L^1}\mathcal{M}(f)(x). \tag{2.5.2}$$

Proof. For a simple function of the form

$$L = \sum_{j=1}^{M} c_j \chi_{B(0,r_j)} = \sum_{i=0}^{M-1} (c_{i+1}+\cdots+c_{M-1}+c_M)\chi_{B(0,r_{i+1})\setminus B(0,r_i)} \tag{2.5.3}$$

with $c_j > 0$, $r_0 = 0 < r_1 < \cdots < r_M$, for $x \in \mathbf{R}^n$ and $t > 0$ we have

$$(L_t * |f|)(x) = \sum_{j=1}^{M} c_j |B(0,r_j)| \frac{(\chi_{B(0,tr_j)} * |f|)(x)}{|B(0,tr_j)|} \le \big\|L\big\|_{L^1}\mathcal{M}(f)(x). \tag{2.5.4}$$

But an arbitrary nonnegative radially decreasing function on $\mathbf{R}^n$ can be pointwise approximated by an increasing sequence of functions L^k of the form (2.5.3). We then apply (2.5.4) to each L^k and take the limit as $k \to \infty$ applying the LMCT. Finally, taking the supremum over all $t > 0$, we deduce (2.5.2). □

The following is an application of Theorem 2.5.1.

Proposition 2.5.4. *The space*

$$\big\{\varphi \in \mathscr{S}(\mathbf{R}^n) :\ \widehat{\varphi} \in \mathscr{C}_0^\infty \textit{ and } \widehat{\varphi} \textit{ vanishes in a neighborhood of } 0\big\}$$

is dense in $L^p(\mathbf{R}^n)$ for $1 < p < \infty$.

Proof. Start with a $\mathscr{C}_0^\infty$ function $\widehat{\Phi}$ which is equal to 1 on the unit ball and vanishes outside the ball $B(0,2)$. Consider the family $1 - \widehat{\Phi}(\xi/\varepsilon)$ which converges pointwise to 1 for $\xi \neq 0$ and the family $\widehat{\Phi}(\varepsilon\xi) \to 1$ as $\varepsilon \to 0$. Then $\big(1 - \widehat{\Phi}(\xi/\varepsilon)\big)\widehat{\Phi}(\varepsilon\xi)$ converges pointwise to $\chi_{\mathbf{R}^n\setminus\{0\}}(\xi)$ for all $\xi \in \mathbf{R}^n$ and vanishes for $|\xi| \ge 2/\varepsilon$ and for $|\xi| \le \varepsilon$. Let $h \in \mathscr{S}(\mathbf{R}^n)$. Applying the LDCT we obtain

$$\lim_{\varepsilon\to 0}\int_{\mathbf{R}^n} \widehat{h}(\xi)\Big(1-\widehat{\Phi}\big(\tfrac{\xi}{\varepsilon}\big)\Big)\widehat{\Phi}(\varepsilon\xi)e^{2\pi i x\cdot\xi}\,d\xi = \int_{\mathbf{R}^n}\widehat{h}(\xi)e^{2\pi i x\cdot\xi}\,d\xi = h(x)$$

for any $x \in \mathbf{R}^n$. In other words the sequence $h_\varepsilon = h * \Phi_\varepsilon - h * \Phi_{1/\varepsilon} * \Phi_\varepsilon$ converges pointwise everywhere to h and its Fourier transform has compact support and vanishes in a neighborhood of the origin. Moreover, for some constant C_Φ we have

[3] Such a function is said to be *radially decreasing*.