$$\frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{q} + m - 1.$$

Prove the inequality

$$\big\| f_1 * \cdots * f_m \big\|_{L^q(\mathbf{R}^n)} \le \big\| f_1 \big\|_{L^{p_1}(\mathbf{R}^n)} \cdots \big\| f_m \big\|_{L^{p_m}(\mathbf{R}^n)}.$$

2.4.3. (Schur's test) Let (X,μ), (Y,ν) be σ-finite measure spaces and let $K(x,y)$ be a nonnegative measurable function on $X \times Y$. Define

$$S(f)(y) = \int_X K(x,y) f(x)\, d\mu(x)$$

for a nonnegative measurable function f on X. Assume that

$$A_0 = \operatorname*{ess.sup}_{y\in Y} \int_X K(x,y)\, d\mu(x) < \infty, \qquad A_1 = \operatorname*{ess.sup}_{x\in X} \int_Y K(x,y)\, d\nu(y) < \infty.$$

Show that S is well defined on $L^1(X) + L^\infty(X)$ and that it maps $L^p(X)$ to $L^p(Y)$ with bound at most $A_0^{1-\frac{1}{p}} A_1^{\frac{1}{p}}$ for $1 < p < \infty$.

2.5 Approximate Identities and Almost Everywhere Convergence

In this section we use the Hardy–Littlewood maximal operator to pointwise control averages with respect to approximate identities. As a result, we deduce almost everywhere convergence properties for approximate identities convolved with certain locally integrable functions.

For an integrable function K on $\mathbf{R}^n$ we define the L^1 dilations K_t of K by setting $K_t(x) = t^{-n} K(x/t)$ for $x \in \mathbf{R}^n$ and $t > 0$.

Theorem 2.5.1. *Let K in $L^1(\mathbf{R}^n)$ satisfy $|K(x)| \le A\,|x|^{-n} \min(|x|^\gamma, |x|^{-\gamma})$, $A, \gamma > 0$, and let f be in $L^1_{\mathrm{loc}}(\mathbf{R}^n)$. Then for some constant $C_{n,\gamma} < \infty$ and all $x \in \mathbf{R}^n$ we have*

$$\sup_{t>0} (|f| * |K_t|)(x) \le C_{n,\gamma} A\, \mathcal{M}(f)(x). \tag{2.5.1}$$

Proof. We have

$$\begin{aligned}
(|f| * |K_t|)(x) &\le \frac{A}{t^n} \int_{\mathbf{R}^n} |f(x-y)| \left|\frac{y}{t}\right|^{-n} \min\left(\left|\frac{y}{t}\right|^{\gamma}, \left|\frac{y}{t}\right|^{-\gamma} \right) dy \\
&= \frac{A}{t^n} \sum_{k=-\infty}^{\infty} \int_{2^k t < |y| \le 2^{k+1} t} \min\left(\left|\frac{y}{t}\right|^{\gamma}, \left|\frac{y}{t}\right|^{-\gamma} \right) \left|\frac{y}{t}\right|^{-n} |f(x-y)|\, dy \\
&\le \frac{A}{t^n} \sum_{k=-\infty}^{\infty} \min(2^{(k+1)\gamma}, 2^{-k\gamma}) \frac{1}{2^{kn}} \int_{2^k t < |y| \le 2^{k+1} t} |f(x-y)|\, dy
\end{aligned}$$