

Then for all $f \in L^p(\mathbf{R}^n)$ we have

$$\|f * g\|_{L^q(\mathbf{R}^n)} \leq \|f\|_{L^p(\mathbf{R}^n)} \|g\|_{L^r(\mathbf{R}^n)}. \quad (2.4.12)$$

Proof. On $L^1 \cup L^{r'}$ define the operator $T(f) = f * g$. By Minkowski's inequality $T : L^1 \rightarrow L^r$ with norm at most $\|g\|_{L^r}$. By Hölder's inequality $T : L^{r'} \rightarrow L^\infty$ with norm at most $\|g\|_{L^r}$. Theorem 2.4.1 gives that T has a bounded extension from L^p to L^q with norm at most $\|g\|_{L^r}^\theta \|g\|_{L^r}^{1-\theta} = \|g\|_{L^r}$, where

$$\frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{r'} \quad \text{and} \quad \frac{1}{q} = \frac{1-\theta}{r} + \frac{\theta}{\infty}.$$

If we eliminate θ , these equations reduce to (2.4.11). This completes the proof of (2.4.12); see Figure 2.1. $\square$

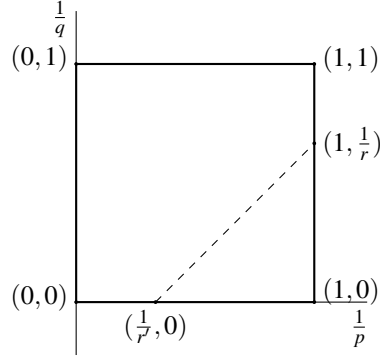


Fig. 2.1 Boundedness on the dotted line is obtained by interpolation between the endpoints $(1/r', 0)$ and $(1, 1/r)$.

Notice that Proposition 2.4.2 is also valid when $r = 1$ (in which case $p = q$) and when $r = \infty$ (in which case $p = 1$ and $q = \infty$) and both of these endpoint cases are just a restatement of Minkowski's convolution inequality (Theorem 1.6.6).

It turns out that for $g \in L^{r,\infty}$ we can obtain the stronger conclusion $\|f * g\|_{L^q(\mathbf{R}^n)} \leq \|f\|_{L^p(\mathbf{R}^n)} \|g\|_{L^{r,\infty}(\mathbf{R}^n)}$ interpolating between $L^1 \rightarrow L^{r,\infty}$ [Exercise 1.6.7 (b)] and $L^{r',1} \rightarrow L^\infty$ (duality between $L^{r',1}$ and $L^{r,\infty}$) using off-diagonal Marcinkiewicz interpolation ([31, Theorem 1.4.19]).

Example 2.4.3. On the real line consider the linear operator

$$L(g)(x) = \int_{2x}^{2x+1} g(x-t) \chi_{[0,1]}(t) dt, \quad x \in \mathbf{R}.$$

We claim that L maps $L^p(\mathbf{R})$ to $L^{p/2}(\mathbf{R})$ for all $1 \leq p \leq \infty$. Obviously L maps L^∞ to itself. If we show that it also maps $L^1(\mathbf{R})$ to $L^{1/2}(\mathbf{R})$, then the conclusion for $p \in (1, \infty)$ will be a consequence of Theorem 2.4.1.

To achieve this we observe that the intersection of the intervals $[2x, 2x+1]$ and $[0, 1]$ is nonempty if and only if $-\frac{1}{2} \leq x \leq \frac{1}{2}$. Thus, for every integrable function g , $L(g)$ is always supported in the interval $[-\frac{1}{2}, \frac{1}{2}]$. Next we notice that

$$\int_{\mathbf{R}} |L(g)(x)| dx \leq \int_{\mathbf{R}} \int_0^1 |g(x-t)| dt dx = \|g\|_{L^1}, \quad (2.4.13)$$

that is, L maps $L^1(\mathbf{R})$ to itself with bound 1. Since $L(g)$ is supported in $[-\frac{1}{2}, \frac{1}{2}]$, an applying the Cauchy–Schwarz inequality and (2.4.13) gives

$$\|L(g)\|_{L^{1/2}(\mathbf{R})} = \|L(g)\|_{L^{1/2}([-\frac{1}{2}, \frac{1}{2}])} \leq \|L(g)\|_{L^1} \leq \|g\|_{L^1}.$$

Thus L maps L^1 to $L^{1/2}$ with bound 1. Combining this with the trivial estimate that L maps L^∞ to itself with bound 1, we deduce that, for $1 < p < \infty$, one has

$$\|L(g)\|_{L^{p/2}} \leq \|g\|_{L^p},$$

by applying Theorem 2.4.1.

Proposition 2.4.4. (Hausdorff–Young inequality) *Let $1 \leq p \leq 2$. Then for every function f in $L^p(\mathbf{R}^n)$ we have the estimate*

$$\|\widehat{f}\|_{L^{p'}} \leq \|f\|_{L^p}. \quad (2.4.14)$$

Proof. We apply Theorem 2.4.1 to interpolate between the estimates

$$\|\widehat{f}\|_{L^\infty} \leq \|f\|_{L^1}$$

[Estimate (2.1.2) in Proposition 2.1.5] and

$$\|\widehat{f}\|_{L^2} \leq \|f\|_{L^2}$$

to obtain (2.4.14). We conclude that, when $1 \leq p \leq 2$, the Fourier transform is a bounded operator from $L^p(\mathbf{R}^n)$ to $L^{p'}(\mathbf{R}^n)$ with norm at most 1. $\square$

Exercises

2.4.1. Let $1 \leq r \leq p \leq 2 \leq q \leq \infty$ be related as in $1/p + 1/q = 1/r$ and fix K in $L^r(\mathbf{R}^n)$. Prove that the linear operator $f \mapsto \widehat{f} * K$ maps $L^p(\mathbf{R}^n)$ to $L^q(\mathbf{R}^n)$.

2.4.2. Let $m \geq 3$ and $1 \leq p_1, p_2, \dots, p_m, q \leq \infty$ be related as in

$$\frac{1}{p_1} + \dots + \frac{1}{p_m} = \frac{1}{q} + m - 1.$$

Prove the inequality

$$\|f_1 * \dots * f_m\|_{L^q(\mathbf{R}^n)} \leq \|f_1\|_{L^{p_1}(\mathbf{R}^n)} \cdots \|f_m\|_{L^{p_m}(\mathbf{R}^n)}.$$

2.4.3. (Schur's test) Let (X, μ) , (Y, ν) be σ -finite measure spaces and let $K(x, y)$ be a nonnegative measurable function on $X \times Y$. Define

$$S(f)(y) = \int_X K(x, y) f(x) d\mu(x)$$

for a nonnegative measurable function f on X . Assume that