As the part of the integral in (2.3.7) over the region $1 \le |x| \le \max(|\xi|^{-1},1)$ produces a constant, we focus on the part over $\max(|\xi|^{-1},1) < |x| \le N$ [for $N > \max(|\xi|^{-1},1)$]. Expressing this part of the integral in (2.3.7) in terms of polar coordinates and inserting (2.3.8), we reduce to the existence of the limit

$$\lim_{N\to\infty}\int_{\max(|\xi|^{-1},1)}^{N} r^{z+n-1}\left[\frac{e^{2\pi i r|\xi|}e^{-i\frac{\pi(n-1)}{4}}+e^{-2\pi i r|\xi|}e^{i\frac{\pi(n-1)}{4}}}{r^{\frac{n-1}{2}}|\xi|^{\frac{n-1}{2}}}+\frac{2\pi R(r|\xi|)}{r^{\frac{n-2}{2}}|\xi|^{\frac{n-2}{2}}}\right]dr$$

as $N\to\infty$. In view of the bound $|R(r|\xi|)| \le C(r|\xi|)^{-3/2}$, the part of the integral containing $R(r|\xi|)$ converges absolutely as long as $\operatorname{Re} z+n-1-\frac{3}{2}-\frac{n-2}{2} < -1$, i.e., $\operatorname{Re} z < -\frac{n-1}{2}$, so for this part the limit exists. In the part of the integral containing the exponentials we write $e^{\pm 2\pi i r|\xi|} = (\pm 2\pi i|\xi|)^{-1}\frac{d}{dr}e^{\pm 2\pi i r|\xi|}$ and integrate by parts to deduce that the limit exists if $\operatorname{Re} z+n-1-1-\frac{n-1}{2} < -1$, i.e., $\operatorname{Re} z < -\frac{n-1}{2}$ as well. This argument shows that the Fourier transform of F_z^2 can be defined pointwise at every point $\xi \in \mathbf{R}^n\setminus\{0\}$, thus $E_{F_z^2} = \{0\}$ using the notation of Proposition 2.3.6.

As $F_z \in L^1+L^2$, we wish to evaluate $\widehat{F_z}$. We begin with the observation that

$$F_z(\lambda x) = \lambda^z F_z(x) \qquad \text{for any } \lambda>0 \text{ and } x\in\mathbf{R}^n\setminus\{0\}.$$

Applying the Fourier transform in this identity and using Proposition 2.1.6 (7) and Proposition 2.3.6 (6) (combined with the fact $E_{F_z^2} = \{0\}$) we obtain that

$$\lambda^{-n}\widehat{F_z}(\lambda^{-1}\xi) = \lambda^z\widehat{F_z}(\xi) \qquad \text{for any } \lambda>0 \text{ and } \xi\in\mathbf{R}^n\setminus\{0\}.$$

This implies that $\widehat{F_z}$ is homogeneous of degree $-n-z$, i.e., $\widehat{F_z}(\lambda\xi) = \lambda^{-n-z}\widehat{F_z}(\xi)$ for all λ and $\xi\in\mathbf{R}^n\setminus\{0\}$.

In view of Corollary 2.1.7 for F_z^1 and Proposition 2.3.6 (8) for F_z^2 (which uses that $E_{F_z^2} = \{0\}$), we obtain that $\widehat{F_z}$ is a radial function, i.e., it has the form $g(|\xi|)$ for some function g on the line. Then for $|\xi|\neq 0$ we have

$$\widehat{F_z}(\xi) = |\xi|^{-n-z}\widehat{F_z}(\xi/|\xi|) = |\xi|^{-n-z}g\big(|\xi/|\xi||\big) = |\xi|^{-n-z}g(1).$$

It could be the case that $|g(1)| = \infty$, but in this case $\widehat{F_z}$ would equal infinity at every nonzero point, and thus it could not belong to $L^\infty(\mathbf{R}^n)+L^2(\mathbf{R}^n)$. We conclude that for some finite nonzero constant $c(z,n)$ we have

$$\widehat{F_z}(\xi) = c(z,n)|\xi|^{-n-z} \qquad\qquad \text{for } \xi\in\mathbf{R}^n\setminus\{0\}. \tag{2.3.9}$$

Finally, notice that as $-n/2 < -\operatorname{Re} z - n < 0$, the part $c(z,n)|\xi|^{-n-z}\chi_{|\xi|<1}$ of $\widehat{F_z}(\xi)$ lies in $L^2(\mathbf{R}^n)$ and the part $c(z,n)|\xi|^{-n-z}\chi_{|\xi|\ge 1}$ is bounded.