which implies

$$\operatorname{Im}\int_{\mathbf{R}^n} f(x)\overline{g(x)}\,dx = \operatorname{Im}\int_{\mathbf{R}^n} \widehat{f}(\xi)\overline{\widehat{g}(\xi)}\,d\xi$$

as $\operatorname{Im} w = \operatorname{Re}(-iw)$. Thus we deduce (ii).

As $\widehat{\phi_N} \to \widehat{f}$ in L^2, it follows that $(\widehat{\phi_N})^\vee \to (\widehat{f})^\vee$ in L^2. Since $\phi_N \in \mathscr{S}(\mathbf{R}^n)$ we have $(\widehat{\phi_N})^\vee = \phi_N$, which converges to f in $L^2(\mathbf{R}^n)$. It follows that f and $(\widehat{f})^\vee$ are equal in $L^2(\mathbf{R}^n)$ and consequently equal almost everywhere. This proves (iii). To prove (iv) we simply take $\overline{g} = \widehat{h}$ (equivalently $h = \overline{\widehat{g}}$) in identity (ii). □

Example 2.3.8. We estimate the Fourier transform of the function $g(t) = t^{-\gamma}\chi_{t\ge 1}$ for $1/2 < \gamma < 1$. This function lies in $L^2(\mathbf{R})$ but does not lie in $L^1(\mathbf{R})$. If we can show that the limit

$$\lim_{N\to\infty}\int_1^N \frac{e^{-2\pi i t\xi}}{t^\gamma}\,dt$$

exists for all $\xi \neq 0$, then $\widehat{g}(\xi)$ can be identified with this limit. We make a few observations. When $\xi = 0$, this limit is infinite, so it is expected that $\widehat{g}(\xi)$ gets worse as $\xi \to 0$. We observe that $|\widehat{g}(\xi)| = |\widehat{g}(|\xi|)|$ as $\widehat{g}(\xi) = \overline{\widehat{g}(|\xi|)}$ for $\xi < 0$. Thus we may work with $|\xi|$ instead of $\xi \neq 0$. A change of variables gives

$$\widehat{g}(|\xi|) = \frac{1}{|\xi|^{1-\gamma}}\lim_{N\to\infty}\int_{|\xi|}^N \frac{e^{-2\pi i t}}{t^\gamma}\,dt = \frac{1}{|\xi|^{1-\gamma}}\left[\frac{e^{-2\pi i|\xi|}}{2\pi i|\xi|^\gamma} - \frac{\gamma}{2\pi i}\int_{|\xi|}^\infty \frac{e^{-2\pi i t}}{t^{\gamma+1}}\,dt\right],$$

where the second identity follows by an integration by parts and evaluation of the limit. From this we obtain that $\widehat{g}(\xi)$ exists when $\xi \neq 0$ and satisfies

$$|\widehat{g}(\xi)| \le \frac{1}{\pi|\xi|} \qquad \text{for } |\xi| \ge 1, \tag{2.3.4}$$

in fact $|\widehat{g}(\xi)| \approx |\xi|^{-1}$ as $|\xi| \to \infty$. For $0 < |\xi| < 1$, writing

$$\widehat{g}(|\xi|) = \frac{1}{|\xi|^{1-\gamma}}\left[\int_{|\xi|}^1 \frac{e^{-2\pi i t}}{t^\gamma}\,dt + \lim_{N\to\infty}\int_1^N \frac{e^{-2\pi i t}}{t^\gamma}\,dt\right], \tag{2.3.5}$$

we deduce

$$|\widehat{g}(\xi)| \le \frac{1}{|\xi|^{1-\gamma}}\left[\frac{1-|\xi|^{1-\gamma}}{1-\gamma} + \frac{1}{\pi}\right] \qquad \text{for } 0 < |\xi| < 1.$$

We conclude

$$|\widehat{g}(\xi)| \le \frac{(1-\gamma)^{-1} + \pi^{-1}}{|\xi|^{1-\gamma}} \qquad \text{for } 0 < |\xi| < 1. \tag{2.3.6}$$

Estimates (2.3.4) and (2.3.6) explain why $\widehat{g} \in L^2(\mathbf{R})$.

Having set down the basic facts concerning the action of the Fourier transform on L^1 and L^2, we extend its definition on $L^1 + L^2$, which in particular contains L^p for $1 < p < 2$.