Here we used the notation $-E_f = \{-\xi : \xi \in E_f\}$, $y+E_f = \{y+\xi : \xi \in E_f\}$, $\lambda E_f = \{\lambda \xi : \xi \in E_f\}$, $A^t E_f = \{A^t\xi : \xi \in E_f\}$.

Proof. To prove (1), we notice that $\widehat{(f+g)_N} = \widehat{f_N} + \widehat{g_N}$ converges to $\widehat{f}+\widehat{g}$ on the complement of the null set $E_f \cup E_g$, therefore $\widehat{f+g}$ lies in $L^2_{\text{ae}}(\mathbf{R}^n)$ and equals $\widehat{f}+\widehat{g}$ on $\mathbf{R}^n \setminus (E_f \cup E_g)$. Moreover $E_{f+g} \subsetneqq E_f \cup E_g$. To prove property (2) we note that $\widehat{(bf)_N} = \widehat{bf_N} = b\widehat{f_N}$ but $b\widehat{f_N}$ converges to $b\widehat{f}$ on $\mathbf{R}^n \setminus E_f$. Thus bf lies in L^2_{ae} and $\widehat{bf} = b\widehat{f}$. For (3) observe that $\widehat{(\widetilde{f})_N} = \widehat{\widetilde{f_N}} = \widetilde{\widehat{f_N}}$ and that $\widetilde{\widehat{f_N}} \to \widetilde{\widehat{f}}$ on the complement of $-E_f$. Hence so does $\widehat{(\widetilde{f})_N}$, thus $\widetilde{f}$ lies in L^2_{ae} and $\widehat{\widetilde{f}} = \widetilde{\widehat{f}}$ on $\mathbf{R}^n \setminus (-E_f)$. For (4) notice that conjugating $\widehat{f_N}(\xi) \to \widehat{f}(\xi)$ yields $\widehat{(\overline{f})_N}(-\xi) \to \overline{\widehat{f}}(\xi)$ when $\xi \notin E_f$, thus $\overline{f}$ lies in L^2_{ae} and the identity in (4) holds on the complement of $-E_f$. For (5) we observe that $(e^{2\pi i(\cdot)\cdot y} f_N)\widehat{\ }(\xi) = \widehat{f_N}(\xi - y)$ which converges to $\widehat{f}(\xi - y)$ when ξ lies in $y+E_f$. Property (6) is proved similarly except that the translation is replaced by dilation. The proof of (7) relies on the identity $\widehat{f_N \circ A} = \widehat{f_N} \circ A$ [Proposition 2.1.6 (9)] and the fact that the balls $B(0,N)$ remain invariant under rotations. To prove (8), we note that as f is defined on $\mathbf{R}^n \setminus \{0\}$, then we have $f = f \circ A$ on $\mathbf{R}^n \setminus \{0\}$ for every orthogonal matrix A. The fact that $E_f = \{0\}$ yields that $A^t E_f = \{0\}$; on the complement of this set we have the identity $\widehat{f \circ A} = \widehat{f} \circ A$ by part (7). But $f = f \circ A$ implies $\widehat{f} = \widehat{f \circ A}$ and $\widehat{f \circ A} = \widehat{f} \circ A$ for every orthogonal matrix A, we deduce that $\widehat{f}$ is radial. □

Proposition 2.3.7. *For* f, g, h *in* L^2 *we have*

(i) ***(Plancherel's identity)*** $\|f\|_{L^2(\mathbf{R}^n)} = \|\widehat{f}\|_{L^2(\mathbf{R}^n)}$

(ii) ***(Parseval's identity)*** $\displaystyle\int_{\mathbf{R}^n} f(x)\overline{g(x)}\,dx = \int_{\mathbf{R}^n} \widehat{f}(\xi)\overline{\widehat{g}(\xi)}\,d\xi$

(iii) ***(Fourier inversion)*** $(\widehat{f})^\vee = \widehat{f^\vee} = f$ *a.e.*

(iv) ***(Jumping hat identity)*** *We have* $\displaystyle\int_{\mathbf{R}^n} f(x)\widehat{h}(x)\,dx = \int_{\mathbf{R}^n} \widehat{f}(\xi)h(\xi)\,d\xi$.

Proof. Given $f \in L^2(\mathbf{R}^n)$, pick a sequence of Schwartz functions ϕ_N (which certainly lie in $L^1 \cap L^2$) such that $\phi_N \to f$ in $L^2(\mathbf{R}^n)$. Proposition 2.3.1 gives that $\|\phi_N\|_{L^2} = \|\widehat{\phi_N}\|_{L^2}$ for all N. As the definition of $\widehat{f}$ is independent of the sequence converging to the function, $\widehat{\phi_N} \to \widehat{f}$ in $L^2(\mathbf{R}^n)$. Thus $\|\phi_N\|_{L^2} \to \|f\|_{L^2}$ and $\|\widehat{\phi_N}\|_{L^2} \to \|\widehat{f}\|_{L^2}$ as $N \to \infty$ and thus assertion (i) follows.

To prove (ii) we use *polarization* as follows: We apply (i) to $f+g$, we expand both sides, and we use (i) for f and g to obtain

$$\operatorname{Re}\int_{\mathbf{R}^n} f(x)\overline{g(x)}\,dx = \operatorname{Re}\int_{\mathbf{R}^n} \widehat{f}(\xi)\overline{\widehat{g}(\xi)}\,d\xi.$$

We then apply (i) to $f+ig$ and likewise we obtain

$$\operatorname{Re}\int_{\mathbf{R}^n} -if(x)\overline{g(x)}\,dx = \operatorname{Re}\int_{\mathbf{R}^n} -i\widehat{f}(\xi)\overline{\widehat{g}(\xi)}\,d\xi$$