(b) Prove that $\{Q_m\}_{m=1}^{\infty}$ is an approximate identity on $\mathbf{R}$ as $m \to \infty$.
(c) Given a continuous function $f$ on $\mathbf{R}$ supported in $[0,1]$, show that

$$c_m \int_0^1 f(t)(1-(x-t)^2)^m \, dt = (f * Q_m)(x), \qquad x \in [0,1],$$

is a sequence of polynomials that converges to $f$ uniformly on $[0,1]$ as $m \to \infty$.

**1.9.4. (Fejér kernel)** Show that the sequence of functions $\{F_m\}_{m=1}^{\infty}$ defined on the real line by

$$F_m(t) = \frac{1}{m+1} \left| \frac{\sin(\pi(m+1)t)}{\sin(\pi t)} \right|^2 \chi_{|t| \le \frac{1}{2}} = \frac{1}{m+1} \left| \sum_{k=0}^{m} e^{2\pi i k t} \right|^2 \chi_{|t| \le \frac{1}{2}}$$

is an approximate identity as $m \to \infty$. Also verify the preceding identity.

**1.9.5. (Continuous Fejér kernel)** Show that the family $\{F_\varepsilon\}_{\varepsilon > 0}$ defined on $\mathbf{R}^n$ by

$$F_\varepsilon(x_1, \dots, x_n) = \frac{1}{\varepsilon^n} \prod_{j=1}^{n} \left( \frac{\sin(\pi x_j/\varepsilon)}{\pi x_j/\varepsilon} \right)^2$$

forms an approximate identity. $\big[$*Hint:* Use Exercise 2.3.3.$\big]$

**1.9.6.** Let $0 < b < 1$ and $z$ be a complex number with $|z| < 1/b$. Let $f$ be a continuous function on the line. Prove that

$$\lim_{m \to \infty} \frac{1}{b^m} \sum_{j=m}^{\infty} z^{j-m} \int_{b^{j+1}}^{b^j} f(t) \, dt = \frac{1-b}{1-bz} f(0).$$

$\big[$*Hint:* Apply Theorem 1.9.7 for a suitable approximate identity sequence $\{K_m\}_{m=1}^{\infty}$.$\big]$

**1.9.7.** Let $\Omega$ be an open subset of $\mathbf{R}^n$. Denote by $\mathscr{C}_0^\infty(\Omega)$ the space of all smooth functions with compact support contained in $\Omega$. Suppose that $f, g$ are locally integrable functions on $\Omega$ (i.e., they lie in $L^1(K)$ for any compact subset $K$ of $\Omega$). Assume that

$$\int f(x)\varphi(x) \, dx = \int g(x)\varphi(x) \, dx \qquad \text{for all } \varphi \text{ in } \mathscr{C}_0^\infty(\Omega).$$

Prove that $f = g$ a.e. on $\Omega$. $\big[$*Hint:* It will suffice to prove $f = g$ a.e. on every ball $B(x_0, r)$ contained in $\Omega$. Let $\varepsilon_0 = \operatorname{dist}(\overline{B(x_0,r)}, \Omega^c)$ (or $\varepsilon_0 = 1$ if $\Omega = \mathbf{R}^n$). Pick a smooth function $\varphi$ with integral 1 supported in the unit ball of $\mathbf{R}^n$ and define $h = f - g$ on $\overline{B(x_0, r + \frac{\varepsilon_0}{3})}$ and zero elsewhere. Then $h \in L^1(\mathbf{R}^n)$ and $\varphi_\varepsilon * h$ is supported in $\overline{B(x_0, r + \frac{2\varepsilon_0}{3})}$ and vanishes on $\overline{B(x_0, r)}$ for $\varepsilon < \varepsilon_0/3$. Apply Theorem 1.9.4 (a).$\big]$