A simple modification in the proof of Theorem 1.9.4 yields the following variant.

Theorem 1.9.7. *Let K_δ be a family of functions on $\mathbf{R}^n$ that satisfies properties* (i) *and* (iii) *of Definition 1.9.1 and also*

$$\int_{\mathbf{R}^n} K_\delta(y)\,dy = A \tag{1.9.6}$$

for some fixed $A \in \mathbf{C}$ and for all $\delta > 0$.

(a) *If $f \in L^p(\mathbf{R}^n)$ for some $1 \le p < \infty$, then $\|K_\delta * f - Af\|_{L^p(\mathbf{R}^n)} \to 0$ as $\delta \to 0$.*
(b) *If f in $L^\infty(\mathbf{R}^n)$, then $\|K_\delta * f - Af\|_{L^\infty(E)} \to 0$ as $\delta \to 0$, provided f is uniformly continuous in a neighborhood of a subset E of $\mathbf{R}^n$ in the sense of* (1.9.1).

A family of functions $\{K_\delta\}_{\delta>0}$ that satisfies properties (i) and (iii) of Definition 1.9.1 and also (1.9.6) for some $A \neq 0$ is called an A-multiple of an approximate identity. In the case where $A = 0$, it is called *an approximate zero family*.

As an application of the notion of approximate identities we show that $\mathscr{C}_0^\infty(\mathbf{R}^n)$ is a dense subspace of $L^p(\mathbf{R}^n)$ for all $1 \le p < \infty$.

Example 1.9.8. Given $f \in L^p(\mathbf{R}^n)$ and $\varepsilon > 0$ we find a compactly supported function h such that $\|f-h\|_{L^p(\mathbf{R}^n)} < \varepsilon/2$. In fact such an h can be chosen to be $f\chi_{|f|<M}$ for some large M (since $f\chi_{|f|<M} \to f$ in $L^p(\mathbf{R}^n)$ as $M \to \infty$ by the LDCT). Next we find a compactly supported smooth function K on $\mathbf{R}^n$ with integral 1 and we consider the approximate identity $\{K_\delta\}_{\delta>0}$. Then in view of Theorem 1.9.4, there is a $\delta > 0$ such that $\|K_\delta * h - h\|_{L^p(\mathbf{R}^n)} < \varepsilon/2$. It follows that $\|K_\delta * h - f\|_{L^p(\mathbf{R}^n)} < \varepsilon$ and notice that $K_\delta * h$ is both smooth and compactly supported.

Exercises

1.9.1. Show that for all $x \in \mathbf{R}$ we have

$$\lim_{\varepsilon\to 0^+} \varepsilon \int_{\mathbf{R}} \frac{y\cos(\sin(x-y))}{(y^2+\varepsilon^2)^{3/2}}\,dy = 0\,.$$

Moreover, the convergence is uniform in x on the real line.

1.9.2. For $m = 1,2,\dots$ let B_m be balls in $\mathbf{R}^n$ that contain the origin and whose measures shrink to 0 as $m \to \infty$. Prove that the family of functions $|B_m|^{-1}\chi_{B_m}$ is an approximate identity. Write $B_m = B_m^+ \cup B_m^-$, where B_m^+, B_m^- are disjoint and equimeasurable. Show that the sequence $|B_m|^{-1}\chi_{B_m^+} - |B_m|^{-1}\chi_{B_m^-}$ is an approximate zero family as $m \to \infty$.

1.9.3. Let $Q_m(t) = c_m(1-t^2)^m$ for $t \in [-1,1]$ and zero elsewhere, where c_m is a constant chosen such that $\int_{-1}^1 Q_m(t)\,dt = 1$ for all $m = 1,2,\dots$.
(a) Prove that $c_m \le (m+1)/2$. $\big[$*Hint:* Use $(1-t^2)^m \ge (1-|t|)^m$ when $|t| \le 1$.$\big]$