Next we define convergence on the space of Schwartz functions

Definition 1.8.5. Let $\{f_j\}_{j=1}^\infty$ be a sequence of Schwartz functions. We say that f_j converges to a Schwartz function f *in the Schwartz topology,* or simply *in* $\mathscr{S}(\mathbf{R}^n)$, if $\rho_{\alpha,\beta}(f_j - f) \to 0$ as $j \to \infty$ for all multi-indices α, β. We then write $f_j \to f$ in $\mathscr{S}$.

In particular, if $f_j \to f$ in $\mathscr{S}$ as $j \to \infty$, then for all multi-indices β, the sequence $\partial^\beta f_j - \partial^\beta f$ tends to zero uniformly on $\mathbf{R}^n$.

Example 1.8.6. The sequence of Schwartz functions $f_j(x) = e^{-1/x}e^{-jx}\chi_{(0,\infty)}$ on the real line converges to zero in $\mathscr{S}(\mathbf{R})$ as $j \to \infty$. To verify this assertion, first we notice that for each $m \in \mathbf{Z}^+$ there is a polynomial P_m of degree $2m$ such that

$$\frac{d^m}{dx^m}(e^{-\frac{1}{x}}) = P_m(\tfrac{1}{x})e^{-\frac{1}{x}},$$

a fact that will be tacitly used in the sequel. Now for $j \geq 1$ and for nonnegative integers K, L we estimate

$$\begin{aligned}
&\rho_{K,L}(f_j)\\
&\quad \leq \sum_{l=0}^{L}\binom{L}{l} j^{L-l}\sup_{x\geq 1}\left|x^K e^{-jx}\frac{d^l}{dx^l}(e^{-\frac{1}{x}})\right| + \sum_{l=0}^{L}\binom{L}{l} j^{L-l}\sup_{0\leq x<1} e^{-jx}\left|\frac{d^l}{dx^l}(e^{-\frac{1}{x}})\right|.
\end{aligned}$$

The first supremum tends to zero as $j \to \infty$ since $e^{-jx} \leq e^{-j/2}e^{-x/2}$ when $j, x \geq 1$. In the second supremum notice that the lth derivative of $e^{-1/x}$ on $[0,1)$ is bounded by $C_M x^M$ for any $M \in \mathbf{Z}^+$. Choosing $M = L+1$ we bound the second term by

$$C_L' j^L e^{-jx}x^{L+1} \leq \frac{C_L'}{j}\sup_{t>0}(t^{L+1}e^{-t}),$$

which also tends to zero as $j \to \infty$.

Theorem 1.8.7. *The space* $\mathscr{C}_0^\infty(\mathbf{R}^n)$ *is dense in* $\mathscr{S}(\mathbf{R}^n)$ *in the Schwartz topology. Precisely, fix a smooth function* φ *with values in* $[0,1]$ *supported in* $B(0,2)$ *and equal to* 1 *on the unit ball* $B(0,1)$*. Then for any* $f \in \mathscr{S}(\mathbf{R}^n)$*, the sequence* $f_j(x) = f(x)\varphi(x/j)$ *converges to* $f(x)$ *in the Schwartz topology as* $j \to \infty$*.*

Proof. For fixed multi-indices α and β we show that $\rho_{\alpha,\beta}(f\varphi(\cdot/j) - f)$ tends to zero as $j \to \infty$. By Leibniz's rule we estimate this Schwartz seminorm by

$$\sum_{\substack{\gamma\leq\beta\\ \gamma\neq\beta}}\binom{\beta}{\gamma}\frac{1}{j^{|\beta|-|\gamma|}}\sup_{x\in\mathbf{R}^n}\left|x^\alpha(\partial^\gamma f)(x)(\partial^{\beta-\gamma}\varphi)\Big(\frac{x}{j}\Big)\right| + \sup_{x\in\mathbf{R}^n}\left|x^\alpha(\partial^\beta f)(x)\Big(\varphi\Big(\frac{x}{j}\Big)-1\Big)\right|.$$

As $(\partial^{\beta-\gamma}\varphi)(x/j)$ remains bounded for all j, the first term tends to zero as $j \to \infty$, since $|\beta| - |\gamma| \geq 1$. As $\varphi(x/j) - 1 = 0$ $|x| < j$, the second supremum equals