1.8 Schwartz Functions

For a pair of multi-indices α and β and a function $f \in \mathscr{C}^\infty(\mathbf{R}^n)$ we define the $\rho_{\alpha,\beta}$ *Schwartz seminorm*[7] of f by

$$\rho_{\alpha,\beta}(f) = \sup_{x \in \mathbf{R}^n} |x^\alpha \partial^\beta f(x)|.$$

Naturally, this quantity could be infinite for certain smooth functions.

Definition 1.8.1. A $\mathscr{C}^\infty$ complex-valued function f on $\mathbf{R}^n$ is called a Schwartz function if for all multi-indices α and β we have $\rho_{\alpha,\beta}(f) < \infty$. The space of all Schwartz functions on $\mathbf{R}^n$ is denoted by $\mathscr{S}(\mathbf{R}^n)$.

Thus a $\mathscr{C}^\infty$ function is called Schwartz if and only if for every multi-index β and every $N \in \mathbf{Z}^+$ there is a constant $C_{N,\beta}$ such that for all $x \in \mathbf{R}^n$ we have

$$|\partial^\beta f(x)| \le \frac{C_{N,\beta}}{(1+|x|)^N}.$$

Obviously, every smooth function with compact support is a Schwartz function, i.e., $\mathscr{C}_0^\infty(\mathbf{R}^n)$ is contained in $\mathscr{S}(\mathbf{R}^n)$.

Example 1.8.2. The function $e^{-|x|^2}$ lies in $\mathscr{S}(\mathbf{R}^n)$ but $e^{-|x|}$ does not, since the latter fails to be differentiable at the origin. The $\mathscr{C}^\infty$ function $g(x) = (1+|x|^2)^{-10}$ is not in $\mathscr{S}(\mathbf{R}^n)$, as $\rho_{\alpha_1,0}(g) = \infty$ for $\alpha_1 = (21,0,\dots,0)$ and $0 = (0,\dots,0)$.

Example 1.8.3. The function $e^{-1/x}e^{-x}\chi_{(0,\infty)}$ lies in $\mathscr{S}(\mathbf{R})$ as $e^{-1/x}\chi_{(0,\infty)}$ is infinitely differentiable at the origin with vanishing derivatives of all orders.

Proposition 1.8.4. *Let f, g be in $\mathscr{S}(\mathbf{R}^n)$ and $c \in \mathbf{C}$. Then $f+g$, cf, fg, and $f*g$ lie in $\mathscr{S}(\mathbf{R}^n)$.*

Proof. The only nontrivial assertion is that $\partial^\beta(f*g)$ has rapid decay at infinity. For each $N > 0$ there are constants $C_{N,\beta}$ and $C'_{N+n+1,0}$ such that

$$\left| \int_{\mathbf{R}^n} (\partial^\beta f)(x-y)g(y)\,dy \right| \le \int_{\mathbf{R}^n} \frac{C_{N,\beta}}{(1+|x-y|)^N} \frac{C'_{N+n+1,0}}{(1+|y|)^{N+n+1}}\,dy. \tag{1.8.1}$$

Inserting the simple estimate $(1+|x-y|)^{-N} \le (1+|y|)^N(1+|x|)^{-N}$ in (1.8.1) we deduce that

$$|(\partial^\beta f*g)(x)| \le C_{N,\beta,n}(1+|x|)^{-N} \int_{\mathbf{R}^n} (1+|y|)^{-n-1}dy = C(N,\beta,n)\,(1+|x|)^{-N},$$

and this proves the rapid decay of $\partial^\beta f*g$ at infinity. □

[7] The Schwartz seminorm is in fact a norm (see Exercise 1.8.2).