Combining these estimates, for $|t| < \delta$, we deduce

$$\big|\Lambda(g,\varphi)(t,x_0)\big| \le \big|\Lambda(g_M,\varphi)(t,x_0)\big| + \big|\Lambda(g-g_M,\varphi)(t,x_0)\big| \le \varepsilon + 2\big\|\partial_j\varphi\big\|_{L^{q'}}\varepsilon.$$

Thus $\Lambda(g,\varphi)(t,x_0)$ converges to zero as $t \to 0$; hence $\partial_j(\varphi * g) = (\partial_j\varphi) * g$ on $\mathbf{R}^n$.

The continuity and boundedness of $\partial_j(\varphi * g)$ is derived from that of $(\partial_j\varphi) * g$ by Theorem 1.6.7. Finally, assuming $\partial^\alpha\varphi \in L^{q'}$ for all $|\alpha| \le m$, we obtain the existence, continuity, and boundedness of $\partial^\alpha(\varphi * g)$ via the identity $\partial^\alpha(\varphi * g) = (\partial^\alpha\varphi) * g$, which is proved by induction on m, applying the case $m = 1$ repeatedly. □

The following corollary shows that the convolution inherits the best degree of smoothness of the two functions.

Corollary 1.7.2. *Let $\varphi \in \mathscr{C}^\infty(\mathbf{R}^n)$, and $g \in L^1(\mathbf{R}^n)$. Assume that $\partial^\alpha\varphi$ lies in $L^\infty(\mathbf{R}^n)$ for all multi-indices α. Then $\varphi * g$ lies in $\mathscr{C}^\infty(\mathbf{R}^n)$.*

Proposition 1.7.3. (a) *There exists a nonnegative and nonzero smooth function supported in a given ball $B(x_0,R)$ in $\mathbf{R}^n$.*
(b) *Given $0 < r_1 < r_2 < \infty$ there exists a smooth function with values in $[0,1]$ supported in $\overline{B(0,r_2)}$ and equal to 1 on $\overline{B(0,r_1)}$.*

Proof. (a) On the real line define the function

$$g(t) = \begin{cases} e^{-1/t} & \text{when } t > 0, \\ 0 & \text{when } t \le 0. \end{cases}$$

Notice that $g \in \mathscr{C}^\infty(\mathbf{R} \setminus \{0\})$, while for $t = 0$ we have

$$\lim_{t\to 0^+}\frac{g(t)-g(0)}{t} = \lim_{t\to 0^+}\frac{e^{-1/t}-0}{t} = 0, \quad \lim_{t\to 0^-}\frac{g(t)-g(0)}{t} = \lim_{t\to 0^-}\frac{0-0}{t} = 0,$$

so $g'(0) = 0$. In a similar way we can show $g''(0) = 0$ and in general $g^{(k)}(0) = 0$ for all k. All these assertions make use of the fact that $\lim_{t\to 0^+} t^{-L}e^{-1/t} = 0$ for any $L > 0$. Then the function $h(t) = g(1-t)g(1+t)$ is smooth, nonnegative, nonzero, and is supported in $[-1,1]$, and so does $h(t^2)$; see Figure 1.4.

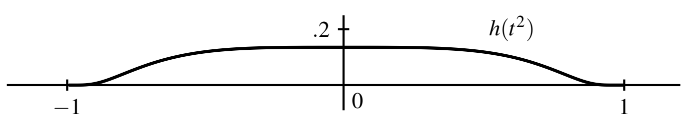

Fig. 1.4 The function $h(t^2)$.

On $\mathbf{R}^n$ consider the function $H(x) = h(|x|^2)$, which is a composition of two smooth functions, hence it is smooth, nonzero, and is supported in the unit ball. Then $H((x-x_0)/R)$ is smooth, nonzero, and supported in the ball $B(x_0,R)$.