$$C_{n,k}(1+|x|)^k \le \sum_{|\beta|\le k} |x^\beta| \le D_{n,k}(1+|x|)^k . \tag{1.7.3}$$

This follows from (1.7.2) for $|x| \ge 1$, while for $|x| < 1$ both terms in (1.7.3) are at least 1 as $|x^{(0,\dots,0)}| = 1$. Here again $0 < C_{n,k} < D_{n,k} < \infty$.

We end the preliminaries by noting the validity of the one-dimensional *Leibniz rule*

$$\frac{d^m}{dt^m}(fg) = \sum_{k=0}^{m} \binom{m}{k} \frac{d^k f}{dt^k} \frac{d^{m-k} g}{dt^{m-k}} , \tag{1.7.4}$$

for all $\mathscr{C}^m$ functions f, g on $\mathbf{R}$, and its multidimensional analog

$$\partial^\alpha (fg) = \sum_{\beta \le \alpha} \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_n}{\beta_n} (\partial^\beta f)(\partial^{\alpha-\beta} g) , \tag{1.7.5}$$

for f, g in $\mathscr{C}^{|\alpha|}(\mathbf{R}^n)$ for some multi-index α, where the notation $\beta \le \alpha$ in (1.7.5) means that β ranges over all multi-indices satisfying $0 \le \beta_j \le \alpha_j$ for all $1 \le j \le n$. We observe that identity (1.7.5) is easily deduced by repeated application of (1.7.4), which in turn is obtained by induction.

Theorem 1.7.1. *Let $m \in \mathbf{Z}^+$, $1 \le q < \infty$, $g \in L^q(\mathbf{R}^n)$, and $\varphi \in \mathscr{C}^m(\mathbf{R}^n) \cap L^{q'}(\mathbf{R}^n)$. Moreover, assume that $\partial^\alpha \varphi$ lies in $L^{q'}(\mathbf{R}^n)$ for all multi-indices α with $|\alpha| \le m$. Then $\varphi * g$ lies in $\mathscr{C}^m(\mathbf{R}^n)$ and $\partial^\alpha(\varphi * g) = (\partial^\alpha \varphi) * g \in L^\infty$ for all $|\alpha| \le m$.*

Proof. Let e_j be the unit vector $(0,\dots,1,\dots,0)$ with 1 in the jth entry and zeros in all the other entries. If $q > 1$ we initially make the additional assumption that g has compact support. If $q = 1$ this initial assumption is not necessary.

We fix an arbitrary $x_0 \in \mathbf{R}^n$ and we show that $\varphi * g$ has a jth partial derivative at x_0, i.e., prove the case $\alpha = e_j$. Using the fundamental theorem of calculus write

$$\begin{aligned} \Lambda(g,\varphi)(t,x_0) &= \frac{(g*\varphi)(x_0+te_j) - (g*\varphi)(x_0)}{t} - g * \partial_j \varphi(x_0) \\ &= \int_0^1 \int_{\mathbf{R}^n} \Big[\big(\partial_j \varphi(y+tse_j) - \partial_j \varphi(y) \big) g(x_0 - y) \Big] \, dy \, ds. \end{aligned} \tag{1.7.6}$$

We note that the integrand in (1.7.6) tends pointwise to zero as $t \to 0$ by the fact that $\varphi \in \mathscr{C}^1$; moreover it is bounded by $2\|\partial_j \varphi\|_{L^\infty(B(x_0,2)-\operatorname{supp} g)} |g(x_0 - \cdot)|$ which lies in $L^1(\mathbf{R}^n \times [0,1], dy\,ds)$. The last assertion follows by the hypotheses of the theorem if $q = 1$ and by the additional assumption that it lies in L^q and is supported in a compact set, on which $\partial_j \varphi$ is certainly bounded, if $q > 1$. The LDCT then yields that $\Lambda(g,\varphi)(t,x_0) \to 0$ as $t \to 0$ for any fixed $x_0 \in \mathbf{R}^n$ when g has compact support.

We now remove the assumption that g has compact support if $q > 1$. Given $g \in L^q$, set $g_M(x) = g(x)\chi_{|x| \le M}$ for $M > 0$. Given a point x_0 and $\varepsilon > 0$ we find an M such that $\|g - g_M\|_{L^q} < \varepsilon$ (as $q < \infty$) and we pick a $\delta > 0$ such that for $|t| < \delta$ we have $|\Lambda(g_M, \varphi)(t,x_0)| < \varepsilon$. Additionally, by Hölder's inequality we obtain

$$\big|\Lambda(g - g_M, \varphi)(t,x_0)\big| \le \int_0^1 \big\|g - g_M\big\|_{L^q} \big\|\partial_j \varphi(\cdot + tse_j) - \partial_j \varphi\big\|_{L^{q'}} \, ds \le 2\varepsilon \big\|\partial_j \varphi\big\|_{L^{q'}} .$$