1.6.5. (a) Let $f \in L^1(\mathbf{R}^n)$ and $g \in L^\infty(\mathbf{R}^n)$ and suppose that g has compact support. Prove that $(f*g)(x) \to 0$ as $|x| \to \infty$.
(b) Provide examples of $f \in L^1(\mathbf{R})$ compactly supported and $g \in L^\infty(\mathbf{R})$ non-compactly supported, such that $|f*g|$ is a constant; hence the assertion in (a) fails.

1.6.6. Let K be a positive integrable function on $\mathbf{R}^n$ and let $1 \le p \le \infty$. Prove that the norm of the operator $T(f) = f*K$ from $L^p(\mathbf{R}^n)$ to itself is equal to $\|K\|_{L^1}$.
$\big[$*Hint:* Clearly, $\|T\|_{L^p \to L^p} \le \|K\|_{L^1}$. Conversely, fix $0 < \varepsilon < 1$ and let N be a positive integer. Let $\chi_N = \chi_{B(0,N)}$ and for any $R > 0$ let $K_R = K\chi_{B(0,R)}$, where $B(x,R)$ is the ball of radius R centered at x. Observe that for $|x| \le (1-\varepsilon)N$, we have $B(0,N\varepsilon) \subsetneqq B(x,N)$; thus $\int_{\mathbf{R}^n} \chi_N(x-y) K_{N\varepsilon}(y)\,dy = \int_{\mathbf{R}^n} K_{N\varepsilon}(y)\,dy = \|K_{N\varepsilon}\|_{L^1}$. Then for $p < \infty$

$$\frac{\big\|K*\chi_N\big\|_{L^p}^p}{\|\chi_N\|_{L^p}^p} \ge \frac{\big\|K_{N\varepsilon}*\chi_N\big\|_{L^p(B(0,(1-\varepsilon)N)}^p}{\big\|\chi_N\big\|_{L^p}^p} \ge \big\|K_{N\varepsilon}\big\|_{L^1}^p (1-\varepsilon)^n .$$

Let $N \to \infty$ first and then $\varepsilon \to 0$. The case $p = \infty$ is straightforward.$\big]$

1.6.7. Let $1 < p < \infty$. (a) Let K be an integrable function on $\mathbf{R}^n$. Show that

$$\big\|f*K\big\|_{L^{p,\infty}} \le \frac{p}{p-1} \big\|K\big\|_{L^1} \big\|f\big\|_{L^{p,\infty}}$$

for all f in $L^{p,\infty}$. Thus the operator $f \mapsto f*K$ maps $L^{p,\infty}(\mathbf{R}^n)$ to $L^{p,\infty}(\mathbf{R}^n)$.
(b) Let $K \in L^{p,\infty}(\mathbf{R}^n)$. Prove that the operator $f \mapsto f*K$ maps $L^1(\mathbf{R}^n)$ to $L^{p,\infty}(\mathbf{R}^n)$ with norm at most $\frac{p}{p-1}\|K\|_{L^{p,\infty}}$.
$\big[$*Hint:* Part (a): Use Theorem 1.2.10. Part (b): Reverse the roles of f and K in (a).$\big]$

1.6.8. Let K be a nonnegative function in $L^1_{\mathrm{loc}}(\mathbf{R}^n)$ and let $0 < p \le \infty$. Suppose that there is a positive constant C such that the inequality

$$\big\|f*K\big\|_{L^{p,\infty}} \le C\big\|f\big\|_{L^p}$$

holds for all nonnegative functions f in $L^{p,\infty}$. Prove that $K \in L^1(\mathbf{R}^n)$. Obtain the same conclusion when $\|f\|_{L^p}$ is replaced by $\|f\|_{L^{p,\infty}}$ in the hypothesis. $\big[$*Hint:* Use that

$$\chi_{B(0,2R)} * K \ge \Big(\int_{B(0,R)} K(x)\,dx\Big)\chi_{B(0,R)}$$

and let $R \to \infty$. Here $B(0,r) = \{x : |x| < r\}$.$\big]$

1.6.9. Let Ω be a measurable subset of $\mathbf{R}^n$ and let $K \ge 0$ be an even measurable function on $\mathbf{R}^n$. Let $T_K(f) = f*K$ for f measurable.
(a) Show that for $1 \le p \le \infty$ we have

$$\big\|T_K\big\|_{L^p(\Omega)\to L^p(\Omega)} \le \big\|F\big\|_{L^\infty(\Omega)}, \quad \text{where} \quad F(x) = \int_\Omega K(x-y)\,dy.$$