It follows from this that

$$\frac{1}{2^n\lambda} w\big(\{x\in Q:\ w>\lambda\}\big) \le |\{x\in Q:\ w>\lambda/[w]_{A_1}\}|,$$

as w is an A_1 weight. In view of (1.2.3), we write

$$\begin{aligned}
\int_Q w^r\,dx &= \int_Q w^{r-1}\,w\,dx \\
&= (r-1)\int_0^\infty \lambda^{r-2} w\big(\{x\in Q:\ w(x)>\lambda\}\big)\,d\lambda \\
&= (r-1)\bigg[\int_0^a + \int_a^\infty\bigg]\lambda^{r-2} w\big(\{x\in Q:\ w(x)>\lambda\}\big)\,d\lambda \\
&\le a^{r-1}w(Q) + (r-1)\int_a^\infty \lambda^{r-2}2^n\lambda\,\big|\{x\in Q:\ w(x)>\lambda/[w]_{A_1}\}\big|\,d\lambda \\
&\le a^{r-1}w(Q) + 2^n[w]_{A_1}^r(r-1)\int_0^\infty \lambda^{r-1}\big|\{x\in Q:\ w(x)>\lambda\}\big|\,d\lambda \\
&= a^{r-1}w(Q) + 2^n[w]_{A_1}^r\frac{r-1}{r}\int_Q w^r\,dx.
\end{aligned}$$

Choosing $a = w_Q = \frac{1}{|Q|}\int_Q w\,dx$ we obtain

$$\Big(1-2^n[w]_{A_1}^r\frac{r-1}{r}\Big)\frac{1}{|Q|}\int_Q w^r\,dx \le \bigg(\frac{1}{|Q|}\int_Q w\,dx\bigg)^r.$$

We now examine for which $r\ge 1$ we have

$$1-2^n[w]_{A_1}^r\frac{r-1}{r} \ge \frac{1}{K^r},$$

or equivalently,

$$2^{\frac{n}{r}}\Big(\frac{r-1}{r}\Big)^{\frac{1}{r}} \le \frac{(K^r-1)^{\frac{1}{r}}}{K}\frac{1}{[w]_{A_1}}. \tag{8.6.4}$$

It may be difficult to exactly determine these r, but as $K>1$, note that (8.6.4) is a consequence of

$$0 \le 2^n e^{\frac{1}{e}}(r-1) \le \frac{K-1}{K}\frac{1}{[w]_{A_1}}, \tag{8.6.5}$$

in view of the inequality $(1-1/r)^{1/r} \le e^{1/e}(r-1)$ for $r\ge 1$. This last inequality can be obtained by setting $s=1-1/r$ and observing that the minimum of s^s over $[0,1]$ is $e^{-1/e}$. Now as (8.6.5) is equivalent to (8.6.3), (8.6.1) is proven. $\square$

Remark 8.6.4. The constant $e^{1/e}$ in (8.6.3) could be replaced by the smallest constant c that satisfies the inequality $(1-1/t)^{1/t} \le c\,(t-1)$ for all $t>1$.

A consequence of the reverse Hölder property of A_p weights is the following characterization of A_1 weights. This provides a converse to Theorem 8.5.3.