Proof. Let us assume $1 \le p < \infty$. The case $p = \infty$ can be handled by reversing the roles of f and g. Given $\varepsilon > 0$, let φ be a continuous function with compact support such that $\|f - \varphi\|_{L^p} < \varepsilon$. Let us suppose that the support of φ is contained in $B(0,M)$. Then φ is uniformly continuous, so there is $\delta > 0$ such that

$$x \in \mathbf{R}^n,\ |h| < \delta \implies \big|\varphi(x+h) - \varphi(x)\big| < \varepsilon |B(0,M+1)|^{-\frac{1}{p}}.$$

For $|h| < \min(\delta, 1)$ Hölder's inequality yields

$$\begin{aligned}\Big|(\varphi * g)(x+h) - (\varphi * g)(x)\Big| &\le \Big[\int_{|y|\le M+1} |\varphi(y+h) - \varphi(y)|^p\, dy\Big]^{\frac{1}{p}} \big\|g\big\|_{L^{p'}} \\ &\le (\varepsilon |B(0,M+1)|^{-\frac{1}{p}}) |B(0,M+1)|^{\frac{1}{p}} \big\|g\big\|_{L^{p'}}.\end{aligned}$$

Then for $|h| < \min(\delta, 1)$ we have

$$\begin{aligned}&\big|(f*g)(x+h) - (f*g)(x)\big| \\ &\qquad \le \big|(\varphi * g)(x+h) - (\varphi * g)(x)\big| + \big|((f-\varphi)*g)(x+h) - ((f-\varphi)*g)(x)\big| \\ &\qquad \le \varepsilon \big\|g\big\|_{L^{p'}} + 2\big\|f - \varphi\big\|_{L^p} \big\|g\big\|_{L^{p'}} \\ &\qquad \le 3\,\varepsilon \big\|g\big\|_{L^{p'}}.\end{aligned}$$

This proves the uniform continuity of $f*g$ on $\mathbf{R}^n$. Its boundedness is a consequence of Hölder's inequality. ☐

Exercises

1.6.1. Show that the support of the convolution of two functions is contained in the in the closure of the algebraic sum[6] of the supports of the two functions.

1.6.2. Let f, g, h be nonnegative measurable functions on $\mathbf{R}^n$ and let $1 \le p < \infty$. Prove that

$$\big((f*g)^p * h\big)^{\frac{1}{p}} \le \min\big[f*(g^p * h)^{\frac{1}{p}}, g*(f^p * h)^{\frac{1}{p}}\big].$$

[*Hint:* Use the Minkowski integral inequality.]

1.6.3. Let $\alpha \in \mathbf{R}^+$ and $\beta \in \mathbf{R}$. Consider the functions $g(t) = e^{-\alpha t}\chi_{t>0}$ and $h(t) = e^{i\beta t}$ defined on the real line. Show that for any positive integer m we have

$$\underbrace{g*\cdots*g}_{m \text{ times}} * h = (\alpha + i\beta)^{-m} h.$$

1.6.4. Consider the Gaussian function $G(x) = e^{-\pi|x|^2}$ on $\mathbf{R}^n$. Show that $(G*G)(x) = G(x/\sqrt{2})/(\sqrt{2})^n$. [*Hint:* Change variables $y = y' + \frac{x}{2}$.]

[6] The algebraic sum of the sets A and B is the set $A + B = \{a+b:\ a \in A, b \in B\}$.