$$\begin{aligned} &= \|f\|_{L^1(\mathbf{R}^n)} \|g\|_{L^1(\mathbf{R}^n)} \\ &< +\infty, \end{aligned}$$

having used Tonelli's theorem. Thus (1.6.1) holds. Since

$$|f * g| \le |f| * |g|,$$

we conclude

$$\big\|f * g\big\|_{L^1(\mathbf{R}^n)} \le \|f\|_{L^1(\mathbf{R}^n)} \|g\|_{L^1(\mathbf{R}^n)} . \tag{1.6.3}$$

Example 1.6.3. On $\mathbf{R}$ consider the convolution of the two characteristic functions $\chi_{[-a,a]}$ and $\chi_{[-b,b]}$, where $0 < a \le b < \infty$. Then for any $x \in \mathbf{R}$ we obtain

$$(\chi_{[-a,a]} * \chi_{[-b,b]})(x) = \big|[-a,a] \cap [-b+x, b+x]\big|$$

and a straightforward calculation yields that

$$\Big|[-a,a] \cap [-b+x, b+x]\Big| = \begin{cases} 2a & \text{when } |x| \le b-a, \\ a+b-|x| & \text{when } b-a < |x| \le a+b, \\ 0 & \text{when } |x| > a+b. \end{cases}$$

The following example indicates how the convolution improves smoothness.

Example 1.6.4. Let $h = \chi_{[-1,1]}$. A calculation gives that $(h*h)(x) = 2-|x|$ for $|x| \le 2$ and $(h*h)(x) = 0$ for $|x| > 2$. Also $(h*h*h)(x)$ equals

$$\begin{cases} 3-|x|^2 & \text{if } |x| \le 1, \\ 4-2|x| + \frac{(|x|-1)^2}{2} & \text{if } 1 < |x| \le 3, \\ 0 & \text{if } 3 < |x|. \end{cases}$$

It turns out that $h*h$ is continuous (but not continuously differentiable) and $h*h*h$ lies in $\mathscr{C}^1$ but not in $\mathscr{C}^2$. The graphs of $h*h*h$ and its derivative are shown in Figure 1.3.

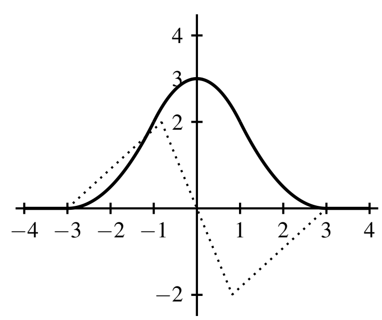

Fig. 1.3 The triple convolution $h*h*h$ and its piecewise linear derivative (dotted) are plotted.

Example 1.6.5. We compute the convolution of $\chi_{[-1,1]}$ and $|x|^{-1}$ on $\mathbf{R}$:

$$h(x) = \big(\chi_{[-1,1]} * |\cdot|^{-1}\big)(x) = \int_{-1+x}^{1+x} \frac{dy}{|y|} = \begin{cases} \log \frac{x+1}{x-1} & \text{if } x > 1, \\ \log \frac{x-1}{x+1} & \text{if } x < -1, \\ \infty & \text{if } |x| \le 1. \end{cases}$$